DOCUMENT RESUME

ED 052 026 SE 011 344

AUTHOR	Titterton, J. Patrick
TITLE	Ten Years of Problems from the NASSAU County Interscholastic Mathematics League 1960-1969 with Answers and Sample Solutions.
INSTITUTION	Syosset Central School District 2, N.Y.
PUB DATE	Jul 69
NOTE	106p.
EDRS PRICE	EDRS Price MF-$0.65 HC-$6.58
DESCRIPTORS	*Mathematical Enrichment, *Mathematics, *Problem Sets, Resource Materials, *Secondary School Mathematics

ABSTRACT

Presented are more than 300 problems of varying difficulty along with answers and solutions. These problems were initially given to participants in mathematics contests among high school students. With each problem a time factor is included as well as a percentage of participants who successfully solved that problem. (CT)

ED052026

U.S. DEPARTMENT OF HEALTH,
EDUCATION & WELFARE
OFFICE OF EDUCATION
THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL OFFICE OF EDUCATION POSITION OR POLICY

TEN YEARS OF PROBLEMS
From the
NASSAU COUNTY INTERSCHOLASTIC MATHEMATICS LEAGUE
1960-1969
With Answers And Sample Solutions

PUBLISHED BY: SYOSSET PUBLIC SCHOOLS

CSD 2

SE 011 344

ED052026

TEN YEARS OF PROBLEMS

from the

NASSAU COUNTY INTERSCHOLASTIC MATHEMATICS LEAGUE

1960 - 1969

With Answers and Sample Solutions

Edited, Produced and Directed By: **J. PATRICK TITTERTON**

Instructor of Mathematics,
Syosset High School

Who has been kicking around as an officer of the

LEAGUE

for six years,

Lately as Executive Secretary

Problems Authored By: **MR. HARRY SITOMER**

Presently Associated With
C. W. Post College

Who is a man of many

and

DIVERSE TALENTS

Sample Solutions Obtained By: **ABSCONDING W. IDEAS**

from
Various and Sundry Individuals

July 6, 1969

FOREWORD

TIME: That is the word. Not enough of it. This booklet has got to be described as a pilot project, because I've got to have some excuse for the many errors you're bound to find in it. I set myself the task of putting it together in just two weeks; of course, I've been involved for seven years, so there is obviously more than just two weeks worth of work in the booklet.

However, what I did have to do was to make up sample solutions, hints or what have you for some 175 problems, plus clean up some 40 others already done. This was quite a chore --- although not an unwelcome one --- and I'm not quite certain that I did such a hot job. The answers in my possession for these problems have been handled at least twice previous to my placing them in this journal; I agreed in my mathematics with all but four of them. Two of them I changed (no I'm not going to tell you which ones) because it looked pretty definite to me that arithmetic was the only difficulty to be overcome.

And I've never been known to make an error in arithmetic. Well, hardly ever.

The other two problems (or was it twenty) I have left as I found them; but my choice for the answer will be found in my sample solution. You'll find it when you get there.

Of course this is excluding the four I couldn't get at all. What I did arrive at, I've presented for your perusal --- plus a few suggestions as to how you might go about finding a valid solution.

Gene Devereaux, former Executive Secretary of the League, is responsible directly for most of the solutions from the 1965 - 1966 school year, although I've embellished and made up some new ones. He is also indirectly responsible for the solutions given for the last two years, since as our consultant on problems he has been sending me sample solutions for all those problems. However, in both cases I am finally responsible for whatever has been presented within these pages, because I've picked and chosen from a multitude of solutions, and the final form and choice has been mine.

In this sense, just let me mention a few of the names who are directly responsible for at least one or more of the sample solutions in this booklet. This is a bit dangerous since I'm bound to leave some people out, but I can specifically point out a solution directly attributable to each of these individuals which was rather unique to each of them. They include, besides Gene Devereaux, Grant Duffrin (our other consultant), Elliot Lamb (President of the League the past three years and Author of the last two sets of Play-Off problems), Tony DiLuna, George Lenchner, Ezra Reed, Bob Nelson, Al Smith and John Titterton.

And there is also one long litany of students who have been responsible for many (if not most) of the more clever and incisive solutions.

By the way, the sample solutions are totally worthless unless the problem is first investigated in depth. They have been designed that way. Unless you're deeply involved in the problem, you won't find anything there of value. These are difficult problems that our author has presented us with, and they will not yield to will 0 the whisp or whim. Concentrated effort would be more like it.

One last warning: I had lost all the answers for the 1965 - 1966 problems, so the ones produced within these pages might not be absolutely. However, I'm sure you'll want to put your trust in me. Ho-Ho-Ho.

What to do if you're really bugged? Drop me a line at Syosset High School. If I can't give you satisfaction, I'll try to find someone who will.

DECIMAL NOTATION: 60.3.4.27 means that the problem in question was first presented in the year 1959 - 1960 (only the Spring of the year has been kept); it was the 3rd problem of that year; the time alloted for it was four (4) minutes; and 27% of the Mathletes attempting it in competition were successful in arriving at a correct solution.

I am missing 12 of the alloted times; two from the 61 year, and ten from the 62 year. You'll just have to use your own imagination in guessing how much time the author alloted for these problems.

On the other hand, the percentage is really significant as far as I'm concerned. This tells you how difficult the problem was, and will either boost your morale when you get a tough problem, or be a blow to your ego when you can't even come close to a solution for an easy one. There should be several in the latter category ---- but I know that some of the high percentage problems were very cleverly guessed at. So don't neglect to build your guessing prowess.

In 1964, 1966, 1967, and 1969 there are some A's instead of percentages; this is because the problem as originally given was A mess. These six problems with the A's have been changed from the originals to a more (hopefully) meaningful form. But only those problems which were really foul-ups have been changed; others which were still meaningful (but not necessarily what the author intended) have been left in their original form.

At any rate, here it is. Your own problem book. No more excuses. You can study our author, Harry Sitomer, at great length. He does bring back the oldies and goodies. I believe if you can handle every problem in this book you've got to be good for at least 15 points a year. The only two students I've known who had done all these problems were first and second in the county. So get the show on the road if you intend to impress as a Mathlete.

As for myself, I expect to be number one in the County this coming year!!!!!

J. Patrick Titterton

July 6, 1969
(Whoops, it's 2AM ; make that:) July 7, 1969

INDEX

FOREWORD p. -2
BACKWORD Same page
PROBLEMS: 1959 - 1960 p. 1
1960 - 1961 p. 3
1961 - 1962 p. 6
1962 - 1963 p. 9
1963 - 1964 p. 12
1964 - 1965 p. 15
1965 - 1966 p. 18
1966 - 1967 p. 21
1967 - 1968 p. 25
1968 - 1969 p. 28
Answers pp. 31 - 33

SAMPLE SOLUTIONS, HINTS OR WHAT HAVE YOU:

1959 - 1960 p. 34
1960 - 1961 p. 37
1961 - 1962 p. 41
1962 - 1963 p. 45
1963 - 1964 p. 49
1964 - 1965 p. 53
1965 - 1966 p. 57
1966 - 1967 p. 62
1967 - 1968 p. 65
1968 - 1969 p. 69

Appendix A: Some History of Success in the NCIML pp. 74 - 75

Appendix B: Play-Off Problems Through the Years

Nassau Play-Off Problems p. 76
Nassau - Suffolk Meet Problems p. 81
Answers p. 84

NCIML PROBLEMS 1959 - 1960

60.1.3.04

How many even multiples of 7 are there between 1 and 1,000,000 that are also squares?

60.2.5.04

A number is one more than its reciprocal. Find the sum of the fourth power of this number and the fourth power of the reciprocal.

60.3.4.27

A man walks one mile east, then 1 mile northeast and then 1 mile north. Find the number of miles between his starting point, and ending point, as the crow flies, to the nearest hundredth of a mile.

60.4.6.08

The roots of the equation

$2x^2 - mx - 8 = 0$ differ by $m-1$. Find two values of m.

60.5.6.03

ABCD is a square. A point, P, within the square is 8 inches from A, 6 inches from B and 5 inches from C. Find the number of square inches in the square drawn on PD as a side.

60.6.5.72

In 1932, a boy was as old as the last two digits of his year of birth. His grandfather was able to say the same thing about himself. How many years older was the grandfather?

60.7.5.64

Eight girls leave a group of boys and girls and then the ratio of boys to girls is 3:1. Thereafter, 20 boys leave and the ratio of the remaining boys to girls changes to 5:3. How many girls were there to begin with?

60.8.6.28

In right triangle ABC, AC = 160", BC = 120", and C is the right angle. The perpendicular bisector of AB starts at P in AB and ends in D, a point in AC. Find the number of inches in PD.

60.9.6.20

For what value of n will the product of the real roots of

$x^{2/n} - 12x^{1/n} + 27 = 0$ be 3?

60.10.7.37

Two circles are concentric. The radius of the smaller one is a inches and that of the larger is $2a$ inches. If the smaller radius is diminished by x inches and the larger increased by x inches, then the area between the circles is doubled. Find the ratio of $x:a$.

60.11.4.33

A book is to have 250 pages that will be numbered with arabic numerals. How many times will the digit "2" be used in numbering these pages?

60.12.6.13

A merchant buys goods at 40% off the list price. He marks his goods so that after giving a discount of $33\frac{1}{3}$% he will still have a profit of 20% of the marked price. What is the ratio of the marked price to the list price expressed with smallest possible positive integers?

60.13.8.26

An isosceles trapezoid has bases of 12 inches and 20 inches respectively, and each arm is 8 inches long. A line segment is drawn parallel to the bases dividing the trapezoid into two trapezoids having equal perimeters. Find the ratio of the area of the original trapezoid to that of the smallest one, in terms of smallest positive integers.

60.14.5.58

Solve for x:

$\log_{10}(x^2 - 14x + 5) = 2.$

60.15.6.34

The sides of a pentagon are prolonged to form a five pointed star whose angles form an arithmetic progression having a common difference of 2 degrees. Find the number of degrees in the smallest of these angles.

60.16.5.42

Suppose that a right angle is divided into 100 equal parts called centiangles and that each centiangle is further divided into 10 equal parts called milliangles. How many centiangles and milliangles are there, to the nearest milliangle, in 35° 20' ?

60.17.6.29

Solve for x, y, and z:

$\frac{x-1}{x-2} + \frac{y-2}{y-3} = 4; \quad \frac{y-2}{y-3} + \frac{z-3}{z-4} = 4;$

$\frac{1}{z-4} + \frac{1}{x-2} = 2 .$

60.18.6.51

Triangle ABC is inscribed in a circle whose center is O. If angle B = 10° and angle C = 40°, find the number of degrees in the angle formed by AO and the bisector of angle A.

60.19.7.13

Find the prime factors of $(x-1)(x-2)(x-3)(x-4) - 24$ whose coefficients are integers.

60.20.8.15

In quadrilateral ABCD, AB = 7 inches, BC = 20 inches, CD = 14 inches, DA = 13 inches and AC = 15 inches. BD intersects AC in E. Find the ratio of BE to ED expressed with smallest possible positive integers.

60.21.5.16

Two clocks both indicate the

true time, now. One gains a second in an hour, and the other loses 3 seconds in 2 hours. In how many days will they again both indicate the true time?

60.22.6.37

A man was twice as old as his wife when he was as old as his wife is now. When she will be as old as he is now, their combined ages will be 112 years. How old is the man?

60.23.6.26

In triangle ABC, AB = AC. Point D is taken in AC and E in AB so that BC = BD = BE. Find the ratio of the number of degrees in angle BDC to that of angle EDA.

60.24.5.41

Assuming that $(a+c)(b+c)(a+b) \neq 0$, and that $b^2 - a^2 = c^2 - b^2$, find the value of $(2/(a+c))-(1/(b+c))-(1/(a+b))$.

60.25.6.13

Quadrilateral ABCD is inscribed in a circle. If AB = 8 inches, AD = 5 inches, BC = 3 inches, and angle BAD = 60°, find the number of inches in DC.

And that's it for 59-60; there were only 25 problems given during the first five years of the league.

NCIML PROBLEMS 1960 - 1961

61.1.5.32

A man travels a distance at the rate of 5 miles per hour and returns at the rate of 5 hours per mile. What is his average rate in miles per nour?

61.2.6.23

From D in hypotenuse AB of triangle ABC, perpendiculars are drawn to AC, meeting it in E, and to BC, meeting it in F. If DECF is a square and AC:CB = 2:1, then DB:AB = 1:x. Find x.

61.3.6.20

A set of n numbers has the sum s. Each number is decreased by 5, then multiplied by 5, then increased by 5. The sum of the new set of numbers is then as + bn. Find the numerical value of a+b.

61.4.5.11

In trapezoid ABCD, base AB is 15' long and base DC is 6' long. Point E is a trisection point of diagonal AC, nearer A, and point F is a trisection point of BD nearer B. Find the length, in feet, of EF.

61.5._.28

Given that $a_0 = 2$ and $a_1 = 4$,

and that $a_{n+2} = (3a_{n+1} - 2a_n)/3$,

find the numerical value of a_3.

61.6.6.07

A number of the form abcabc, in which a, b, and c represent digits, is the product of 5 primes. One of these is 491. Find the number.

61.7.9.09

Three circles are externally tangent to each other. Their radii are respectively $(3)^{1/2}+1$, $(3)^{1/2}-1$, and $3-(3)^{1/2}$ inches. The area bounded by the circles is $2(3)^{1/2}-\pi(10-x(3)^{1/2})/3$ square inches. Find x.

61.8.4.92

Take my age 3 years hence and triple it. From this product, subtract 3 times my age 3 years ago, and then you will know how old I am. How old am I?

61.9.7.37

In the power a^b, the base and the exponent are both squared. The new power may then be written as

$((a^b)(a^b))^k$. Express k as a function of b.

61.10.7.10

Equilateral triangle EBF is inscribed in square ABCD, with E in AD and F in DC. The side of the triangle measures 8". The side of the square measures $2(\sqrt{x}+\sqrt{2})$ inches. Find x.

61.11.6.09

Five clubs meet on January 1. Thereafter, the first club meets every second day; the second meets every third day; the third every fourth day; the fourth every fifth day; and the fifth every sixth day. On how many occasions in the next 100 days will exactly 3 clubs meet on the same day?

61.12.6.21

If $x+y = A$ and $xy = B$, then

$(1/x^3) + (1/y^3) = (A^3-k)/B^3$. Express k in terms of A.

61.13.6.43

One third of a container of water is removed and replaced by wine. One third of the mixture is then removed and again replaced by wine. This is done a third and a fourth time. What fractional part of the final mixture is water?

61.14.6.07

In triangle ABC, point D is taken in BC so that BD:DC = 1:2, and E in AC so that AE:EC = 3:1. AD and BE intersect in F. If DF:FA = 1:x, find x.

61.15.5.35

Find the numerical value of $(\log_k 125)(\log_{25} k)(\log_k k^4)$,

where $k \neq 0$.

61.16.5.20

A magician had a magic purse which doubled the amount of money put into it. He agreed to let a greedy man use the purse if the magician was to receive \$32 each time the purse was used. After the fifth doubling the greedy man had no money left. How many dollars did he start with?

61.17.6.05

ABCD is a trapezoid. The coordinates of A are (1,2), B(7,10) and D(5,-1). The bases are AB and DC and C is above the x-axis. The area of the trapezoid is 62.5 square units. Find the coordinates of C.

61.18.6.17

The length of chord AC in circle O is 2" and that of AD is 5". Perpendiculars are drawn to diameter AB from C and D meeting AB in points F and G respectively. If GF = 3", find the number of inches in AB.

61.19.5.05

Find the prime factors of:

$x^4 - 11x^2 + 1$, having rational coefficients.

61.20.5.09

Find, correct to the nearest hundredth,

$$\frac{\sqrt{8 + 2\sqrt{15}} - \sqrt{8 - 2\sqrt{15}}}{\sqrt{8 + 2\sqrt{15}} + \sqrt{8 - 2\sqrt{15}}}$$

(using only positive square roots).

61.21.5.09

Find the integer, z, if 88 times z results in the number 2y831x in which x and y represent digits.

61.22.5.25

Find the polynomial of lowest degree, which, when divided by x^2-9, $x^2+6x-27$, or $x^2+12x+27$ leaves a remainder of 1. Express your answer as the sum of terms.

61.23.6.17

In triangle ABC, D, E, and F are midpoints of sides AB, AC, and BC, respectively. M is the midpoint of DF, and N is the midpoint of EF. Find the ratio of the area of AMN to that of ABC with smallest possible integers.

61.24.5.11

A germ culture increased its population in 1 hour by x%. In the next hour it lost x%. The net result was a loss of 36%. Find x.

61.25.6.06

Through point D, taken in side AC of triangle ABC, a line is drawn parallel to BC, meeting AB in E. DE is then prolonged through E to F so that the length of FE is twice that of ED. FC intersects AB in G. If EA = 24" and BG:GE = 2:3, find the number of inches in AB.

61.26.5.10

A square is divided into 81 equal small squares. One and only one of a set of even consecutive integers beginning with 4 is written in each of the small squares. It is then found that the sums of the integers in each row are equal. What is this sum?

61.27.6.05

From a point whose coordinates are (4,5) a path is drawn to a point on the y-axis, then to a point on the x-axis, and finally to (11,3). What is the length of this path if it is the shortest possible path?

61.28.-.21

ABCD is a square whose side measures 4'. Points E, F, G, and H are the midpoints of AB, BC, CD, and DA respectively. Semicircles are drawn outside the square on AE, BF, CG, and DH, as diameters. The least possible length of a belt around those semicircles is $4\sqrt{m} + n\pi$ feet. Find m+n.

61.29.6.22

Solve for all real values of x:

$$(x^2-3.5x+2.5)^2-3.5(x^2-3.5x+2.5) + 2.5 = 0 .$$

61.30.5.25

Two boys run in opposite directions around a rectangular field whose area is 1700 square yards. They start from the same corner at the same time and meet 10 yards from the opposite corner. If the ratio of their speeds is 4:5, find the perimeter of the field, in yards.

NCIML PROBLEMS 1961 - 1962

62.1.5.36

In this multiplication of a 2-digit number by a 1-digit number, each letter represents a different digit: (I)(DA) = DDD. What is the digit represented by I ?

62.2.5.11

Given that $(a + (1/a))^2 = 3$, where a is a non-zero number, find the numerical value of $a^3 + 1/a^3$.

62.3.6.45

In right triangle ABC, the legs AC and BC are 3" and _" long respectively. The bisector of angle C intersects the hypotenuse in point D. Find the number of square inches in triangle BCD.

62.4.6.33

Given that 3x-4 varies directly as y+5 and inversely as the square of z-5. If y=13 and z=2, then x=10. Find x if y=0 and z=4.

62.5.7.06

Line segment AB is 6" long. P is to be located in a plane containing AB. The distance from P to A is not to be greater than the distance from A to B. Also, the distance from P to B is not to be greater than the distance from B to A. The area of the region in which P may be found is $a\pi - b\sqrt{3}$ square inches. Find the value of a+b.

62.6.4.37

Two unequal numbers have the same pair of digits. The sum of these numbers diminished by their difference is 106. What is the smaller number?

62.7.-.03

One non-zero root of

$$\sqrt[5]{(1+x)^2} + 2\sqrt[5]{(1-x)^2} = 3\sqrt[5]{1-x^2}$$

is n/33. Find n.

62.8.6.28

In parallelogram ABCD, E is the trisection point of AD nearer A, and AC intersects EB is G. If the area of ABCD is 144 square inches, find the number of square inches in AEG.

62.9.-.06

Express as a binomial, the product:

$$(2^{2^0}+1)(2^{2^1}+1)(2^{2^2}+1)\cdots(2^{2^n}+1).$$

62.10.7.04

Arc AD in circle O is a quadrant. Square EFGH is inscribed in this quadrant with the point E on AO, F on OD, and G and H on arc AD. If the radius of the circle is 10" long, find the number of square inches in the area of the square.

62.11.-.52

Write as a simple fraction:

$$\cfrac{\cfrac{\frac{2}{3}+\frac{3}{4}}{38\frac{63}{99}-38\frac{27}{99}}}{\cfrac{62\frac{3}{9}-60\frac{1}{3}}{\frac{81}{99}-\frac{5}{11}}}$$

62.12.-.63

Solve for real values of x:

$$4^{2x} + 2^{2x+1} = 80 .$$

62.13.6.02

A semicircle is drawn on AB as diameter. Chord DC is parallel to AB with C nearer to A than B. If the length of chord CD is twice the length of chord AC and the radius of the semicircle is 5" long, find the number of inches in the length of CD, to the nearest tenth.

62.14.7.25

A grocer uses a balance which is rigged so that the distances of the fulcrum to the 2 pans are unequal. A quantity of flour is placed in the right pan balancing a 1-lb weight in the left pan. Then a quantity of flour is placed in the left pan that balances the 1-lb weight in the right pan. The sum of both quantities of flour is 2 1/12 lb. What is the ratio of the distances from fulcrum to pans?

62.15.5.10

Point C is taken on line segment AB so that the length of BC is twice that of CA. Semicircles are drawn on the same side of AB with CA, CB, and AB as diameters. The number of square inches in the region bounded by the 3 semi-circles is equal to the number of inches in the sum of the lengths of the 3 semicircles. Find the number of inches in AB.

62.16.5.35

If a 2-digit number is multiplied by the product of its digits, a 3 digit number is formed, each digit of which is the unit digit of the original number. Find the original number.

62.17.-.53

Simplify to an integer:

$(2^{n+2} - 2^n + 3)/(2^n + 1)$

62.18.5.65

What is the maximum number of points of intersection when 2 circles and 4 straight lines intersect each other?

62.19.-.20

If $(x+1)^2$ is greater than $5x-1$ and less than $7x-3$, find the integral value of x.

62.20.6.26

Equilateral triangle ABC is inscribed in a circle. D is a point in minor arc BC. The length of chord BD is 3" and the length of chord DC is 5". How many inches are there in the length of AD ?

62.21.5.24

Find the 2 prime factors of 4,891 .

62.22.-.19

If a, b, and c are positive numbers, find the positive square root of:
$(a^2+ab+bc+ca)(b^2+bc+ca+ab)(c^2+ca+ab+bc)$.

62.23.8.03

A diagonal of a regular pentagon is 2'. Find, to the nearest hundredth of a foot, the length of 1 of its sides.

62.24.-.004

If all letters represent real numbers and if:
$(4x^2+9y^2+z^2)(a^2+b^2+c^2) = (2ax+3by+cz)^2$, then $x/3a = y/q = z/r$. Find q+r in terms of b and c.

62.25.-.18

Point A has coordinates (2,-3). Point B has coordinates (4,2). Find the coordinates of point C such that C is on the line AB, B is between A and C, and the length of BC is to the length of AC as 4:5.

62.26.4.13

Given 3 numbers. To the average of 2 of them is added the third. The 3 possible sums are 23, 27, 34. What are the numbers ?

62.27.7.28

Is $3^{48}-1$ exactly divisible by 7? by 11? by 73? In your answer use "1" to mean "yes" and "0" to mean "no". Your complete answer will be a 3-digit representation in which the digit at the left answers the first question, etc.

62.28.6.17

Given triangle ABC with CD the altitude on AB and O the center of the circumscribed circle. If the degree measure of angle A = 82 and that of angle B = 48, find the degree measure of angleDCO.

62.29.-.16

Find the fractional equivalent to $\frac{1}{\sqrt[3]{3} - 1}$ whose denominator is rational.

62.30.7.18

The number of square inches in the area of square ABCD is $8 + 4\sqrt{3}$. Point E is in BC and F is in CD. AEF is an equilateral triangle. Find the number of inches in EF.

NCIML PROBLEMS 1962 - 1963

63.1.5.25

The product of five prime positive integers is a six digit number, each digit of which is equal to the others. Find the number.

63.2.6.29

In the sequence of numbers, $u_1, u_2, u_3, \ldots, u_n$, the value of $u_n = 2u_{n-1} + a$ when $n > 1$. If $u_2 = 5$ and $u_5 = 33$, find a.

63.3.6.49

Angle C of triangle ABC is also an angle of a square and one of the vertices of the square is a point of side AB. The ratio of the length of AC to the length of a side of the square is 5:2. The ratio of the length of AC to the length of BC is m:2. Find m.

63.4.6.25

If $ax + by = m$ and $bx - ay = n$, and $a^2+b^2 = 1$, express x^2+y^2 in terms of m and n.

63.5.5.46

The vertices of triangle ABC are located respectively at (-1,2), (3,7), (4,0). The medians of triangle ABC have point G in common. Find the coordinates of G.

63.6.4.09

What is the greatest integer, which when divided into 383, 527 or 815 leaves the same remainder ?

63.7.6.05

Mr. Jones sold Mr. Smith an article at a profit of a% and later bought it back, giving Mr. Smith a profit of a%. The basis for calculating profits is the cost to the seller. Find a if the net rate of loss to Mr. Jones is 17½% .

63.8.6.46

Each side of triangle ABC measures 24 inches. The perpendicular to side AC, at point D in AC, intersects side AB in E. The perpendicular to AB at E intersects side BC in F. The perpendicular to BC at F intersects AC at D. How many inches are there in segment AD ?

63.9.5.48

A computing machine is set to increase one half of a number fed to it by 10. A positive number, x, is fed to the machine and the result of its operations is then fed back. The second result is found to be x^2. Find x .

63.10.5.54

ABCD is a square. Line AQ intersects side BC. Perpendiculars to AQ from B, C and D meet AQ in F, G and E, respectively. If the length of DE is a and the length of BF is b, express the length of AG in terms of a and b .

63.11.5.42

When a two digit positive integer is multiplied by the number which is tow more than the units'digit, the product has three digits, each of which is the tens' digit of the original number. What is the original number ?

63,12.7.12

One of the roots of $x^3 + ax + b = 0$ is $(4)^{1/3} + (2)^{1/3}$. If a and b are integers, find a.

63.13.6.05

In trapezoid ABCD, leg AD is perpendicular to DC and AC is perpendicular to BD at E. If the ratio of the lengths of AE and DE is 3:4 and the length of AD is 60 inches, find the number of square inches in the area of trapezoid ABCD.

63.14.6.64

For what values of x will the graph of $y = x^2 - x - 2$ be below the graph of $y = x + 1$?

63.15.6.18

Triangle ABC has a right angle at C; the length of AC is 3 inches and the length of BC is 4 inches. Square ABDE and triangle ABC have no interior points in common. The length of CD is $(k)^{1/2}$, where k is an integer. Find k.

63.16.5.03

Find the number of pairs of integers, x and y, such that $x^2 + y^2 < 100$. (An integer may be positive, zero or negative.)

63.17.6.49

Referring to a coordinate system, the distance from (-3,-3) to the line determined by the equation $3x + 2y = -2$, is $k^{1/2}$, where k is an integer. Find k.

63.18.6.32

Each side of square ABCD is 40 miles long. Find the number of square miles within the circle which is tangent to BC and passes through A and D. Express the answer in terms of π.

63.19.6.77

Find real numbers, x and y, such that $3\cdot 2^x = 16y$ and $3^x = 27y$.

63.20.6.02

The altitude of triangle ABC to BC is a feet long. Two lines, parallel to BC, divide the triangle into three regions that have equal areas. The number of feet in the distance between the two lines is ka. Find k to the nearest hundredth.

63.21.5.72

Find the number of integers between 1 and 10,000 that can be expressed in the form $13n^2$ where n is a positive integer.

63.22.5.61

The circumference of a circle exceeds that of a second circle by a. The radius of the first circle exceeds that of the second by ka. Find k to the nearest hundredth.

63.23.6.24

A ship sails due east at the rate of 20 knots. A man on deck, headed northeast, walks at the rate of 10 knots, with respect to the ship. His rate with respect to the sea is

$10\sqrt{a + b\sqrt{2}}$ knots. Find a+b.

63.24.7.29

The lengths of AB and AC, sides of triangle ABC, are 12 inches each, and the length of BC is 8 inches. It is desired to form a second triangle, A'B'C', such that A'B' will be three times as long as AB; B'C' will be three times as long as BC, and A'C' will have the same length as AC. The area of ABC will exceed that of A'B'C' by $k\sqrt{2}$ square inches. Find k.

63.25.6.14

A strip of uniform width is cut from three sides of a square field, leaving a rectangle with an area that is one fourth that of the square. The width of the strip is k times the length of a side of the square. Find k to the nearest hundredth.

63.26.4.46

If the square of a positive number is added to the number, the sum is 145,542. What is the number?

63.27.7.17

The number of square inches in the area of a regular octagon is equal to the number of inches in its perimeter. The radius of the octagon measures $2\sqrt{m - 2\sqrt{2}}$ inches. Find m.

63.28.7.60

In this subtraction, a letter represents one and only one digit.

$$\begin{array}{r} A\,A\,B\,C \\ \underline{B\,C\,A\,A} \\ C\,C\,E\,E \end{array}$$

If C = E − 3, find A.

63.29.7.41

AOB is the quadrant of a circle whose radius is 2a feet and whose center is O. A semicircle is drqwn on OA as diameter inside the quadrant. A circle is then drawn tangent to the semicircle, to the arc of the quadrant, and to OB. Find the length of the radius of this circle in terms of a.

63.30.5.44

Find the coordinates of all points common to the graphs of $y = x^2 - 2x - 3$ and $(y + 4)/(x-1) = -2$.

NCIML PROBLEMS 1963 - 1964

64.1.5.28

Given that 43/30 is written as an equivalent continued fraction $a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d}}}$ where a, b, c, d are positive integers. Find a+b+c+d.

64.2.6.13

The distances from a point in the interior of an equilateral triangle to its sides are 1, 2 and 3 feet respectively. The area of the triangle is $k(3)^{1/2}$ square feet. Find k.

64.3.5.59

Find the value of $-(-x-x^x)^{-x}$ if $x = -2$.

64.4.6.19

In triangle ABC, points D, E and F are taken in sides AB, BC and CA respectively, such that each is a trisection point nearer A, B and C respectively. Segments AE, BF and CD are drawn. How many triangles are there in the diagram?

64.5.6.42

The inequalities $1 < \frac{n-3}{3n} < 3$ are true for all numbers, n, such that $a < n < b$. Find b.

64.6.5.47

The product of 99 and an integer, k, is 50x8x, where x represents a digit in the product. Find the integer k.

64.7.6.16

The total surface area of a cube is 125% more than that of a second cube. The volume of the second cube is x% less than that of the first cube. Find x to the nearest unit.

64.8.7.16

The circle inscribed in triangle ABC is tangent to BC at D. The lengths of sides AB, BC, and CA are 5, 6 and 7 inches respectively. The length of AD is x inches. Find x.

64.9.5.59

Given the following definition of $|x|$, where x is a real number: $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. Solve for x: $x^2 + |x| = 30$.

64.10.6.19

Each of three circles has a radius of 15 feet and each is externally tangent to the others. Each circle is also internally tangent to a fourth circle. The area of the fourth circle is $(a + b\sqrt{3})\pi$ feet. Find a.

64.11.5.40

In a two-dimensional coordinate system, the coordinates of the three vertices of a triangle are (0,0), (10,0), and (2,8) respectively. Find the number of points in the interior of the triangle both of whose coordinates are integers.

64.12.6.23

Two sides of a triangle include an obtuse angle and they are 5 and 6 inches long, respectively. The area of the triangle is 9 square inches. The length of the third side is $k^{1/2}$ inches. Find k.

64.13.5.08

The difference between the roots of $mx^2 + 5x - 6 = 0$ is 1. Find m.

64.14.4.60

The bisectors of the exterior angles at B and C of triangle ABC determine an angle of 50°. Find the number of degrees in one of the obtuse angles formed by the bisectors of angle ABC and angle ACB.

64.15.6.31

Starting with the smallest, arrange the following numbers in order of magnitude:

$(4/3)^{1/2}$, $(.9)^{1/2}$, $(3/2)^{1/3}$, $(7/9)^{1/4}$.

64.16.3.71

Find the average of the abscissas of all the points of $4x^2 + 9y^2 = 36$ which lie in the first and second quadrants.

64.17.6.04

Assume that when $a > 0$, $b > 0$, that $\sqrt{-a} \cdot \sqrt{-b} = -\sqrt{ab}$. Solve for real values of x and y:

$$x^2 + y(xy)^{1/2} = 81$$

$$y^2 + x(yx)^{1/2} = 162$$

Write each pair of roots as an ordered pair of numbers having the form (x,y).

64.18.5.63

The mean (sometimes called the average) of n consecutive integers, starting with n, is 31. Find n.

64.19.7.28

The area bounded by the graphs of $y = \sqrt{36 - x^2}$ and $y^2 = 3x^2$ is $k\pi$ square units. (The length of the side of a unit square is the unit length of the coordinate system.) Find k.

64.20.6.20

The bases AB and CD of trapezoid ABCD are 3 and 4 feet long, respectively. AC and BD intersect at E. The sum of the areas of triangle AEB and triangle CDE is 50 square feet. Find the number of square feet in the area of the trapezoid ABCD.

64.21.6.34

Find the least positive integer n, which is not a multiple of 5, such that when n is divided by 8 the remainder is 5 and when n is divided by 13 the remainder is 7.

64.22.7.11

The lengths of the sides of a triangle are 13, 14, and 15 inches, respectively. The length of the radius of the circle that circumscribes the triangle is k/8 inches. Find k.

64.23.7.04

If a two-digit number is multipoied by 7, the digits in the product have the same sum as those in the two-digit number. How many such two-digit numbers are there ? (The tens digit of a two-digit number may not be zero.)

64.24.5.A

Find all ordered pairs (x,y) such that x and y represent real numbers and $(y - x^2)^2 + (y - x - 2)^2 = 0$.

64.25.7.25

The length of each of three sides of a trapezoid is $k(\sqrt{5} - 1)$ miles. One of the diagonals is 6 miles long and forms two isosceles triangles with the sides of the trapezoid. Find k.

64.26.4.76

A train, 100 yards long, takes 30 minutes to clear a tunnel when going at the average rate of 40 miles per hour. How many yards are there in the length of the tunnel? (1760 yards = 1 mile.)

64.27.6.46

The parabola determined by $y = ax^2 + bx + c$ has 2 as its y-intercept, 3 as its x-intercept, and its turning point has 1 as its abscissa. Find a.

64.28.6.19

Point B is an interior point of line segment AC. Circles are drawn on AB and BC as diameters and one of their common external tangents intersects their common internal tangent at F. If the points of contact of the external tangent are D and E, and BC is 12", the length of BE is 10" and the length of BF is $k^{1/2}$ inches, find k.

64.29.5.36

The roots of $9x^2 + 6x + a = 0$ are proportional to the roots of $x^2 + x + 1 = 0$. Find a.

64.30.7.17

Each side of square ABCD is 30 cm. long. The midpoint of BC is F, and E is in AD such that EA = 10 cm. EB and AF intersect in G. Find the number of square cm. in triangle EFG.

NCIML PROBLEMS 1964 - 1965

65.1.5.39

Missing digits in this multiplication are denoted by x.

$$\begin{array}{r} \mathrm{xxx} \\ \underline{\mathrm{x}7} \\ 3\mathrm{xx}4 \\ \underline{5\mathrm{xxx}} \\ \mathrm{xxxx}4 \end{array}$$

Find the product.

65.2.6.05

Find all positive integers represented by x, if $x^2 + x + 19$ is the square of an integer.

65.3.7.01

In triangle ABC, the lengths of AB, AC, and BC are 13, 14 and 15 inches respectively. The median AD intersects the altitude BE in F. The number of square inches in triangle AFE is k/19. Find k.

65.4.5.19

Let x and y represent real numbers. How many points are there in the graph of $((x-y)(y-2))^2 + ((x+y)(y+2))^2 = 0$.

65.5.4.37

A plane and two points in the plane are given. How many regular pentagons are there in the plane which contain the given points as vertices ?

65.6.5.74

What is the units digit of 127^{39} when it is expressed in decimal form ?

65.7.5.20

Let the coordinates of A, B and C be (2,1), (4,-4) and (-1,-1), respectively. Let X be a fourth point which together with A, B and C, in some order, are vertices of a parallelogram. Find the possible coordinates of X.

65.8.5.58

Solve for x, assuming that $x > 0$ and $x \neq 1$:
$\log_{x^2}(2) + \log_4(x) = 1$.

65.9.5.07

Each side of an equilateral triangle is 6 inches long. A circle, having a radius of 2 inches, is moved outside the triangular region in such a way that it has exactly one point in common with the triangle at all times. Find, to the nearest tenth of a square inch, the area bounded by the path along which the center of the circle may move.

65.10.4.24

Find the real values of x for which:

$$x + 1 = \cfrac{1}{1 + \cfrac{1}{1 + x}}$$

65.11.6.61

Find the remainder when $3^8 + 2 \cdot 6^4 + 2^8$ is divided by 26 .

65.12.6.19

A, B, and C ran a mile (5280') race which A won, beating B by 30 feet and C by 90 feet. By how many feet, to the nearest hundredth, did B beat C ? (Assume that the speed of each is constant.)

65.13.6.20

Point P is outside rectangle ABCD. The distances from P to A, B and C respectively, are 6, 3, and 4 inches. The distance from P to D is $k^{1/2}$ inches. Find k.

65.14.5.38

A computing machine is designed to multiply whatever real number is fed into it by x and to add x to the product. The number 3 is fed into the machine and the number reported by the machine is then fed into the machine, after which the machine reports 39. Find x.

65.15.7.07

The area of triangle ABC is 60 square inches. Points D and E trisect side AC, D being nearer A. The median to BC intersects BD in G and BE in H. Find the number of square inches in triangle BGH.

65.16.5.17

The units digit of a six-digit number is 2. If the units digit is removed to the left of the numeral without changing the relative order of the other digits, the new integer thus formed is one-third of the original integer. What is the original integer?

65.17.6.49

A man has walked three-fifths of the distance across a bridge when he sees a train coming toward him. The speed of the train is 60 miles per hour. By running, the man can reach either end of the bridge just as the train reaches that end. Find how many miles per hour the man can run.

65.18.6.29

The longer base of an isosceles trapezoid is as long as one of its diagonals and the shorter base is as long as its altitude. If the shorter base is 12 inches long, how many inches are in the longer base ?

65.19.6.07

Express z as a function of y if $x^2 + 1 = xy$ and $x^{10} + 1 = x^5z$.

65.20.7.07

A circle having a radius of 4 inches is inscribed in triangle ABC touching AB at D. If DB is 6" and AD is 8", the distance from C to the center of the circle is $k^{1/2}$ inches. Find k.

65.21.5.30

Two candles have the same length. One is consumed uniformly in 4 hours, the other in 5 hours. If the candles are lighted at the same time, in how many minutes will one be four times as long as the other ?

65.22.7.18

Each angle of a rectangle is trisected. The trisectors adjacent to each side intersect at points which are vertices of a rhombus. If the dimensions of the rectangle are a and 2a, the length of each side of the rhombus is $a\sqrt{(k - 2\sqrt{3})/3}$. Find k.

65.23.5.29

What values assigned to a and b make $x^4 + 8x^3 + 8x^2 + ax + b$ the square of a trinomial ?

65.24.5.42

Find two numbers, x and y, whose sum, product, and quotient (x/y) are equal.

65.25.6.02

In triangle ABC the lengths of AB, BC and CA are respectively, 13, 14 and 15 inches. The altitudes of the triangle meet at H and AD is the altitude to BC. The ratio of the lengths of AH:HD is k:5. Find k.

65.26.5.01

By measuring with a yardstick that is $12\frac{1}{2}\%$ too short, profits are 25% of sales. Were the yardstick 10% too short profits would be $x/7$ % of sales. Find x.

65.27.5.78

The diagonals of a rhombus are 10 and 20 feet long. The area of the circle inscribed inside the rhombus is $m\pi$ square feet. Find m.

65.28.5.05

Chord AB of circle O divides the circular region into two parts. Find the ratio of the area of the larger part to that of the smaller part, expressed as a decimal to the nearest tenth, if angle AOB is 120°.

65.29.6.29

Express $(a^2+b^2+c^2)(x^2+y^2+z^2) - (ax+by+cz)^2$ as the sum of the squares of three binomials.

65.30.6.08

If $x^2 - (c-3)x - 3c$ and $x^2 + 2cx + c^2-1$ have a common factor of the form $x+a$ where a is a non-zero rational number, what are the possible values of c ?

NCIML PROBLEMS 1965 - 1966

66.1.6.07

List the pairs of prime numbers between 100 and 200 having the property that the decimal numeral representations of the numbers in each pair have the same three distinct digits in some order.

66.2.6.12

The roots of $x^2 - 5x + 3 = 0$ are r and s. Find the numerical value of $(1/r^3) + (1/s^3)$.

66.3.6.14

The length of median AD in triangle ABC is 7 inches. The lengths of sides AB and AC are 8 and 14 inches, respectively. Find the number of inches in side BC.

66.4.5.27

A man rides to work and walks back along the same route. His average rate for the round trip is one-third of his riding rate. Assuming that his walking and riding rates are uniform, how many times his walking rate is his riding rate ?

66.5.5.34

Two circles are concentric. A chord of the larger circle lies in a tangent of the smaller. The chord is 12 inches long. The area of the region bounded by the the two circles is $k\pi$ square inches. Find k.

66.6.6.28

A crossword puzzle is constructed in a square that has 15 cells in a row and 15 such rows. Among the horizontal words are 12 three-letter words, 10 four-letter words, 9 five-letter words, 6 six-letter words, and finally a certain number of eleven-letter words. Among the vertical words are 8 three-letter words, 14 four-letter words, 12 five-letter words, 5 six-letter words, and a certain number of ten-letter words. How many cells are blanked out ?

66.7.6.A

For what positive integral values of p is $(3p+25)/(2p-5)$ a positive integer ?

66.8.6.11

In convex quadrilateral ABCD diagonals AC and BD determine right angles at E. AE and DE are each 6 inches long, and BE and CE are each 8 inches long. The perpendicular from E to AB is extended to meet DC at F. Find the number of inches in segment DF.

66.9.6.30

For all real values of c, $x^3 - 6x^2 + 7x - 5 = a(x-2)^3 + b(x-2)^2 + c(x-2) + d$. Find the numerical value of c.

66.10.6.28

In plotting a graph of a parabola having an equation of the form $y = ax^2 + bx + c$, a student found the following values of y for uniformly spaced values of x: 462, 562, 670, 780, 894, 1012. Exactly one of these values is incorrect. What should its correct value be ?

66.11.5.13

Consider the set of all four-digit decimal numerals, each having the same four distinct integers in some order. The sum of the numbers represented by these numerals is 193,314. What are the four digits ?

66.12.5.06

Find x in terms of a and b if

$$(ax)^{\log a} = (bx)^{\log b},$$

where each log is expressed to the same base and $ab(a-b) \neq 0$.

66.13.6.03

A satellite moves in an elliptical orbit with the earth (a point) at a focus. The point in the orbit nearest the earth (the perigee) is 50 miles from the earth, and the point furthest from the earth (the apogee) is 450 miles from the earth. Relative to a rectangular coordinate system in which the x-axis contains the earth, apogee, and perigee, an equation of the orbit is $x^2/250^2 + y^2/k^2 = 1$. Find k.

66.14.6.16

When the denominator of $6/(\sqrt{5}+\sqrt{3}+\sqrt{2})$ is rationalized, the resulting fraction is of the form $(a\sqrt{2} + b\sqrt{3} + c\sqrt{30})/2$. Find a+b+c.

66.15.6.40

The shortest side of a triangle is 8 inches long. The longest side of a larger similar triangle is 27 inches long. Two sides of the smaller triangle have the same length as two sides of the larger triangle. How many inches are there in the length of the longest side of the smaller triangle ?

66.16.5.36

As a steadily moving train passes over the junction between two rails, a click is heard. If the number of clicks counted in 15 seconds is the same as the speed of the train in miles per hour, and the lengths of rails are the same, how many feet are there in the length of one rail ?

66.17.6.00

Find the numerical value of $((a+b)(b+c)(c+a))/(abc)$ if $(a+b-c)/c = (b+c-a)/a = (c+a-b)/b$ and $abc \neq 0$.

66.18.7.08

The perimeter of a right triangle, measured in feet, is twice the area of the triangle, measured in square feet. The hypotenuse is 9 feet longer than one of the legs. Find the number of feet in the longer leg.

66.19.6.A

The equation $(c+d\sqrt{3})x^2 - 2x + 7 + 4\sqrt{3} = 0$ has exactly one root, given that the ratio of c/d is not a multiple of $\sqrt{3}$. Find c+d.

66.20.5.53

In right triangle ABC, hypotenuse AB measures 16 inches and side AC measures 8 inches. D and E are in side AB such that CE and CD trisect angle ACB. Find the number of inches in segment DE.

66.21.6.02

Suppose that two different size boxes are available for packing regulation size baseballs; one holding 8 balls, the other 15. What is the largest number of balls that cannot be packed in these boxes if each box that is used is to be filled to capacity ?

66.22.6.20

If $(\sqrt{x+1} + \sqrt{x-1})/(\sqrt{x+1} - \sqrt{x-1}) = a/b$, and $ab \neq 0$, then $x = k/(2ab)$. Express k in terms of a and b.

66.23.5.34

If x, y, and z represent real numbers and $x^2 + y^2 + z^2 = 8x + 4y + 6z - 29$, find the numerical value of $x+y^2+z^3$.

66.24.6.36

The sum of the squares of the first n positive integers is k times the sume of these integers. Express n in terms of k. Note that the sum of the squares of the first n integers is $(n(n+1)(2n+1))/6$.

66.25.8.19

If the lengths of the sides of a triangle are $1, r, r^2$, then r can be any real number between r_1 and r_2. Find the numerical value of r_1+r_2.

66.26.6.12

A two digit numeral represents a prime number whether the base of the numeral system is 8, 10 or 16. Find the decimal numeral representation(s) of the number(s).

66.27.6.01

The area of an isosceles right triangle is $(3 - 2\sqrt{2})a^2$. The perimeter is ka. Find k.

66.28.6.01

Arc AB is a quadrant of a circle with center O. The midpoint C of arc AB is a vertex of an equilateral triangle CDE, with D in radius OA and E in radius OB. The area of the region bounded by OA, arc AB, and OB is πk times the area of triangular region CDE. Find k to the nearest hundredth.

66.29.5.51

Find the numerical value of $(ae + c^2)/(ce)$ if $(a+b)/(b+c) = (c+d)/(d+e) = (e+f)/(a+f)$, where $a+b+c+d+e+f \neq 0$ and $a \neq 0$.

66.30.7.21

The radii of two circles are 2 inches and 5 inches long and their centers are 25 inches apart. A common internal tangent touches one circle at A and the other at B. How many inches are there in segment AB ?

NCIML PROBLEMS 1966 - 1967

67.1.5.35

All pages of a book are numbered by decimal numerals, all of which are formed with 858 digits. How many pages are there in the book ?

67.2.5.02

In the equation in x, $(a-x)(b-x) = c$, $c \neq 0$, the roots can be found by setting each of the factors, a-x or b-x, equal to c. Express a in terms of b and c.

67.3.6.29

The vertices of quadrilateral ABCD have respective coordinates (1,3), (3,8), (7,2) and (4,-4) in a rectangular coordinate system. Find the number of square units in the region defined by ABCD, the unit determined by the coordinate system.

67.4.6.39

Factor over the integral domain: $x^4 - 9x^2 + 12x - 4$

67.5.6.05

ABC is an equilateral triangle. D is a point in line BC, such that B is between D and C. The distances from D to lines AC and AB, respectively, are 7 and 4 inches. The number of square inches in the area of the region of triangle ABC is $k^{1/2}$. Find k.

67.6.6.07

The missing digits in this multiplication are denoted by x. Find the product.

$$\begin{array}{r} 4xx \\ xx \\ \hline 19xx \\ 3x0xx \\ \hline xx0xx \end{array}$$

67.7.6.06

In evaluating each of the following: 1/3, 1/(3+(1/3)), 1/(3+(1/(3+(1/3)))), ... and so on, it is found that successive values are closer and closer to some number. What is that number ?

67.8.7.07

Quadrilateral ABCD is inscribed in a circle. The lengths, in feet, of sides AB, BC, CD, and DA are, respectively, 3,4,5, and 6. Find, to the nearest tenth, the number of feet, in diagonal AC.

67.9.6.39

Express $(7x-1)/(x^2-x-6)$ as the sum of two rational expressions, each having a first degree polynomial as denominator and a constant polynomial as numerator.

67.10.6.26

The legs BC and AC of right triangle ABC are, respectively, 2 inches and $2\sqrt{3}$ inches long. Square ABDE is situated in such a manner that E and C are on opposite sides of line AB. Find the length, in inches, of segment CE, expressed in the form $\sqrt{a+b\sqrt{3}}$, where a and b are integers.

67.11.5.A

What is the smallest positive integer, excluding 2, which when divided by 3,4,5,6,7, or 8, leaves a remainder of 2 ?

67.12.6.18

Suppose $u_n = 2u_{n-1} + 1$, where n is a positive integer and $u_0 = 3$. Express u_a in terms of a.

67.13.7.06

Isosceles trapezoid ABCD is circumscribed about a circle. Leg AD is tangent to the circle at E, and the length of ED is three times the length of AE. If the area of the circular region is divided by the area of the trapezoidal region, the quotient is $\pi\sqrt{k}$. Express k in the form a/b, where a and b are relatively prime positive integers.

67.14.7.28

A dwarf, pursued by a giant, takes 8 steps, while the giant takes 3 steps; but 2 steps of the giant are as long as 11 steps of the dwarf. If the dwarf has a start of 85 of his own steps, how many steps does the giant take to overtake the dwarf ?

67.15.6.07

The lengths of sides AB and AC of triangle ABC are, respectively, 24 inches and 20 inches. The perpendicular to AB at B intersects the perpendicular to AC at C in point E. Segment BE is 7 inches long. How many inches are there in side BC ?

67.16.6.76

If 8 men, working 8 hours a day for 8 days, can produce 8 units of work, how many units will 6 men produce, working 6 hours a day for 6 days, if all men produce at the same rate ?

67.17.6.03

Solve for real values of x, expressing roots to the nearest tenth:
$(6x+7)^{1/3} - (6x-7)^{1/3} = 2$

67.18.8.03

The sides AB, BC, and AC of triangle ABC are respectively, 13 miles, 15 miles, and 14 miles long. A line parallel to line BC cuts AB in D and AC in E. Find the number of miles in the distance between lines DE and BC if the sum of the lengths of segments BD and CE is the same as the length of side BC.

67.19.6.76

In this problem we denote the operation of finding the average (arithmetic mean) between any two numbers, a and b, by "a av b". Find the numerical value of c-a, if (a av b) av c - a av (b av c) = 6.

67.20.6.003

In trapezoid ABCD the lengths, in kilometers, of bases AB and DC are 3 and 10 respectively. Point E is in side AD and F in side BC such that line EF is parallel to the bases and the lengths of segments AE and ED have the ratio 3:4. Find the number of kilometers in segment EF.

67.21.4.47

The least common multiple of two numbers is 216 and their greatest common divisor is 36, but one is not a multiple of the other. What are the two numbers ?

67.22.6.67

Find the price, in cents, of a dozen eggs, if, by selling one more than a dozen for the price of a dozen, actually reduces the price of a dozen by 4¢.

67.23.6.22

In isosceles triangle ABC, point E is a trisection point of base BC, the length of segmentAE is 15 centimeters, and the length of each leg is 17 centimeters. Then the number of centimeters in base BC is $3\sqrt{x}$. Find x.

67.24.7.02

Solve for integral values of x, y, and z, expressing solutions in the form (x,y,z): $z^x = y^{2x}$; $2^z = 2 \cdot 4^x$; and $x+y+z = 16$.

67.25.6.14

The area of the region bounded by a regular polygon of 12 sides is 192 square inches. The number of inches in each side of the polygon is $x\sqrt{2-\sqrt{3}}$ inches. Find x.

67.26.5.71

The fifth power of an integer is 4,xxx,xx1 , where the x's represent missing digits, not necessarily the same. What is the integer ?

67.27.7.37

$A_1A_2A_3A_4A_5A_6$ is a regular hexagon. A point B_1 is in line A_1A_2, with A_2 between A_1 and B_1. The perpendicular to line A_1A_2 at B_1 meets line A_2A_3 at B_2. The perpendicular to line A_2A_3 at B_2 meets A_3A_4 at B_3. The perpendicular to line A_3A_4 at B_3 meets A_4A_5 at B_4, and so on, until the perpendicular to A_6A_1 meets A_1A_2 at B_1. How many inches are there in segment A_1B_1 if the length of A_1A_2 is one inch ?

67.28.8.10

A three digit decimal numeral represents the square of an integer. If the digits are reversed, the new numeral, read in another numeral system, still represents the square of the same integer, and read as a decimal numeral then represents the square of another integer. What is the different base written as a decimal numeral ?

67.29.6.07

A Pythagorean primitive triplet consists of three relatively prime positive integers a, b, c, such that $c^2 = a^2 + b^2$. How many different primitive Pythagorean triplets are there with $a = 105$?

67.30.8.00

Point E is in diagonal CA of parallelogram ABCD, and segment CE is one-fourth as long as CA; F is the midpoint of side BC. Line DE intersects side BC in G. The lengths in inches of segments DE, DF, FG, and AC are respectively 9, 12, 2 and $\sqrt{x}$. Find x.

NCIML PROBLEMS 1967 - 1968

68.1.5.51

Said Mr. Young, "I was born in the 19th century and had my xth birthday in the year x^2". Strangely enough, his great-grandfather was able to make the same statement. In what year was the great-grandfather born ?

68.2.7.00

Solve for real values of x and y, giving solutions in the form (x,y):

$$(x - (1/x))(y - (1/y)) = 4$$

$$(x - (1/y))(y - (1/x)) = 25/6$$

68.3.7.04

Each side of triangle ABC is 6 inches long. Points D, E, and F are, respectively, in AB, BC, and CA, and triangle DEF is also equilateral. If the area of the triangular region DEF is half the area of the triangular region ABC, find, to the nearest .1 inch, the possible lengths of AF.

68.4.5.16

The difference between the square of the arithmetic mean of two numbers and the square of their geometric mean is 25. Find the difference between the arithmetic mean of their squares and the square of their arithmetic mean.

68.5.7.02

Point B is in AC, and the inch measures of AB and BC are 2 and 4 respectively. Squares ABDE and BCFG are drawn on the same side of AC. The circles circimscribing these squares intersect at B and H. The inch-measure, HD + DC = $(1/5)\sqrt{k}$. Find k.

68.6.6.47

Find the positive integral values of x and y such that $x + 2y = 20$, and $13x + 11y$ is a multiple of 17.

68.7.6.19

Find all real values of x such that $\sqrt{12-x} - \sqrt{1+x} < 1$.

68.8.6.02

In a right triangle, the length of the median to the hypotenuse is the mean proportional between the lengths of the legs of the triangle. The quotient of the perimeter of the triangle divided by the length of the hypotenuse is $(a + \sqrt{b})/c$, where a, b, and c are integers. Find a, b, and c.

68.9.7.05

Relative to a rectangular coordinate system, a circle has equation $x^2 + y^2 = 4$. Find one equation for the set of all points on both tangents to the circle that intersect in the point with coordinates (4,0).

68.10.7.01

The medians of a triangle are 39, 42, and 45 inches long. What is the area, in square inches, of the triangular region ?

68.11.6.03

Relative to a space rectangular coordinate system, a sphere has the equation $x^2 + y^2 + z^2 = 9$. How many points are there in the interior of the sphere all of whose coordinates are integers ?

68.12.6.43

Express $x^3 + 9x^2 + 22x + 13$ in the form $a(x+2)^3 + b(x+2)^2 + c(x+2) + d$, where a, b, c, and d are integers.

68.13.6.08

ABCD is a parallelogram. Both AB and AD are longer than 1 inch. Point X is in ray AB, not more than 1/2 inch from B, and point Y is in ray AD, not more than 1/3 inch from D. If z is a point such that AXYZ is a parallelogram, what is the area of the region, to the nearest .01 square inch, in which z may be found, given that the measure of angle A is 60.

68.14.9.02

In triangle ABC, D is the midpoint of AB and E is the trisection point of CB nearer B. The following inch-measures are given: AC = 15, AE = 65/3, CD = 35/2 . Find the area in square inches of the triangular region ABC.

68.15.5.22

The sum of the first n terms of a sequence of numbers is given by the expression $n(4n - 3)$. Find a polynomial in r that expresses the rth term in this sequence.

68.16.6.51

Find the least positive integer divisible by 11 such that when it is divided by 3, 5, and 8, the respective remainders are 2, 1, and 5.

68.17.7.25

Solve for the real values of x: $(x-3)(x-4)(x+1)(x+2) = 24$.

68.18.8.01

Equilateral triangle ABC is inscribed in a circle. Point P is a point of the circle. The inch-measure of AB and PC are 19 and 21 respectively. The area in square inches of the region bounded by ACBP is $k\sqrt{3}$. Find k.

68.19.6.19

Find the prime factors of $(x + y)^5 - x^5 - y^5$, expressing the answer as the product of polynomials in x and y with rational coefficients.

68.20.6.06

The lengths of the sides of a right triangle, measured in inches, are relatively prime integers. The radius of the circle inscribed in the triangle is 3 inches. What are the possible inch-measures of the hypotenuse ?

68.21.6.35

A number n is represented by a decimal numeral that begins and ends with the digit 2, all others being 0. How many zeros are there in the numeral if n is the least such number divisible by 17 ?

68.22.7.05

Solve for all real values of x, y, and z, giving answers in the form (x,y,z):

1.) $x(x+y) + z(x-y) = 4$
2.) $y(y+z) + x(y-z) = -4$
3.) $z(z+x) + y(z-x) = 5$

68.23.6.11

Two circles P_1 and P_2 are concentric. Let AB be a chord of P_1 that intersects P_2 in C and D, such that AB = k(CD). There is no instance of a chord AB such that k = 2. If the radius r_1 pf P_1 is 8, what are the possibilities of radial lengths r_2 of P_2 ?

68.24.6.22

Solve for real values of x:

$$\sqrt{5+\sqrt{x}} - \sqrt{5-\sqrt{x}} = (6)/(\sqrt{5+\sqrt{x}}) \; .$$

68.25.7.06

Lines a, b and c are parallel to each other. The distance from b to a is 2 inches, the distance from a to c is 3 inches. What is the length, to the nearest .1 inch, of each side of an equilateral triangle that has one vertex on each of the three lines ?

68.26.5.26

What is the remainder when 2^{38} is divided by 127 ?

68.27.7.05

Two ferries start from opposite sides of a river at the same time, meeting 720 yards from one shore. After remaining in their slips for the same time, they return, this time meeting 400 yards from the other shore. How wide is the river, in yards ?

68.28.6.30

Semi-circle P is drawn on AB as diameter. Point C is an interior point of AB. Semi-circles R and S are drawn on AC and CB respectively on the same side of AB as P. The perpendicular to AB at C intersects P at D and CD is 8 inches long. The area of the region bounded by P, R, and S is $k\pi$ square inches. Find k.

68.29.6.12

In the equation $(1+\sqrt[3]{2}+\sqrt[3]{4})(x+y\sqrt[3]{2}+z\sqrt[3]{4}) = 1$, x, y, and z represent rational numbers. Solve for x, y, and z giving answers in the form (x,y,z).

68.30.3.23

The hypotenuse of a right triangle is $8\sqrt{2}$ inches long. The length of the median to the hypotenuse is the geometric mean of the lengths of the legs. Find the area of the triangle.

NCIML PROBLEMS 1968 - 1969

69.1.4.41

Express as a power of 3:

$$\frac{3^{3^3}}{(3^3)(3^3)^2(3^3)^3}$$

69.2.6.04

Solve for x: $4^x + 4 = 17\cdot 2^{x-1}$.

69.3.7.08

Each side of a square is 12 inches long. Each vertex of a rectangle is located in a different side of the square. The area of the rectangular region is one-third that of that square region. Find the length in inches of the shorter side of the rectangle. Give answer in the form $a\sqrt{b} - b\sqrt{a}$ where a and b are positive integers.

69.4.7.32

One unit of food A has 700 calorie units, 400 vitamin units and costs $2.00 . One unit of food B has 600 calorie units, 500 vitamin units and costs $3.00. One unit of food C has 400 calorie units, 600 vitamin units and costs $4.00 . How much will it cost, in dollars, if, when buying an integral amount of some of each food, I get exactly 5900 calorie units and 4800 vitamin units ?

69.5.7.02

The circle that circimscribes triangle ABC has a radius of $(1/6)\sqrt{k}$ inches long. Find k if the inch-measure of AB, BC, and CA are respectively 5, 6, and $\sqrt{13}$.

69.6.7.28

A decimal numeral, representing an integer that is divisible by 9, is composed of both digits 2 and 3 but no other digits. What is the smallest integer, greater than 4000, about which this is true ?

69.7.4.06

When a is divided by b, the quotient is c and the remainder is d. When c is divided by e the remainder is f. Express the remainder when a is divided by be. ($be \neq 0$, all letters used represent positive integers.)

69.8.4.21

The medians AD, BE, CF of triangle ABC meet at G. The midpoints of DG, BG, and FG are K, L, and M, respectively. What fractional part of the area of region ABC is the area of region KLM ?

69.9.7.14

Let $f(n) = ((1+\sqrt{5})/2)^n + ((1-\sqrt{5})/2)^n$. Express $f(n+1) - f(n-1)$ in terms of $f(n)$.

69.10.5.36

A circle, whose radius is one inch long, is rolled on the outside of a square in the plane of the square and touching the square at all times. The sides of the square are 5 inches long. Find the area, to the nearest tenth of a square inch, of the region bounded by the square and the path of the center of the circle.

69.11.6.38

The first and last digits of a decimal numeral are 2, all others being 0. What is the smallest number about which this is true if it is greater than 2002 and is divisible by 182 ?

69.12.6.27

Solve for real values of x: $\log_{25}x + \log_{x}25 = 1$.

69.13.5.34

Given a plane and two points in it. How many non-coincident regular n-gons are there in the plane that contain the two points as vertices ? Give your answer in terms of n.

69.14.6.24

Consider the set of all polynomials, $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where the a's are real numbers and x is a real variable. For which of these polynomials is it true that for all x, $P(2x) = 2P(x)$ and $P(3) = 12$.

69.15.6.22

The diagonals of parallelogram ABCD intersect at S. The midpoints of AS, BS, CS, and DS are, respectively, A', B', C', and D'. Find the area, in square inches, of the intersection of the region AB'CD' and BC'DA', if the area of ABCD is 48 square inches.

69.16.6.01

Find the least odd positive integer that has exactly 15 divisors, including 1 and itself.

69.17.6.50

A can do a job in 12 days if B helps him for 7 of those days. But B takes 14 days to do the job if A helps him 7 of those days. If A and B work at steady rates, how many days would it take A alone to do the job ?

69.18.7.54

ABCD is a square, with AB = a inches. Point X is any point in AC and Y is any point in BD. What is the area, in terms of a, of the region that contains all the midpoints of all possible segments XY ?

69.19.6.01

Let ABCDEF be a decimal numeral representing the number m, and DEFABC the decimal numeral representing the number n. Find m if m:n = 4:9.

69.20.7.05

Two circles, one having a 12 inch radius, the other having a 3 inch radius, are tangent externally. Find the number of square inches in the area of the region bounded by their common tangents.

69.21.7.A

When $5x + 2y$ is divisible by a prime number p for certain values of x and y, where x and y are relatively prime to p, so is $7x + 5y$. Find the value of p.

69.22.7.A

The sum of the integers in a set of two or more positive consecutive odd integers is 455. What are all the possible values of the smallest integer in the set ?

69.23.7.01

In triangle ABC, the measure of angle A is 30, the measure of angle C is 60, and AB = 12 inches. Equilateral triangles are drawn on AB, BC, and CA, all of them having no points in common with the interior of triangle ABC. The centroids of the equilateral triangle are D, E, and F respectively. The area, in square inches, of region DEF is $k\sqrt{3}$. Find k.

69.24.6.58

Consider the graph of $2|x| + |y| = 4$, relative to a rectangular coordinate system in which the unit distance is 1 inch. How many square inches are there in the region bounded by the graph ?

69.25.7.09

Assume, in a plane rectangular coordinate system, that the lines with equations $y = 0$ and $y = 2$ are banks of a river and that points A and B have coordinates (0,5) and (12,−2), respectively. What is the x-coordinate of the point on the x-axis at which a bridge is to be located, perpendicular to and between the river banks, so that the direct path from A to the bridge, across the bridge, and then to B, shall have the shortest possible distance ?

69.26.6.08

For what positive integral value of x, other than 1, is $x^4 + 6x^3 + 11x^2 + 8x - 17$ the square of an integer ?

69.27.7.67

For what values of (x,y,z) will $5x+2y+3z$ have a maximum value if (x,y,z) satisfies some three of the following conditions: $2x+y+3z = 10$, $x = 0$, $y = 1$, $z = 2$.

69.28.7.07

Each of two circles, having a radius 6 inches long, passes through the center of the other. The area, in square inches, of the union of the associated circular regions is $a\pi + b\sqrt{3}$, where a and b are integers. Find a+b.

69.29.6.05

If $f(x) = a|x| + b|x+k|$, $f(-1) = 0$ and $f(2) = 0$, $b \neq 0$, then find the value of k.

69.30.6.20

The inch measure of AB of square ABCD is 63. The region ABCD is subdivided into four rectangular regions by two lines, one parallel to AB, the other parallel to BC. If R_A, R_B, and R_C represent the areas of the rectangular regions containing A, B and C, then $R_A : R_B : R_C = 3:4:5$. Find R_D.

ANSWERS TO NCIML PROBLEMS 1960 - 1969

1959 - 1960

60.1	142(?)	60.2	7	60.3	2.41	60.4	6, -10/3
60.5	53	60.6	50	60.7	23	60.8	7
60.9	1/3	60.10	1/2	60.11	106	60.12	9/7
60.13	64/19	60.14	19,-5	60.15	32	60.16	.39c,3m
60.17	3,4,5	60.18	15	60.19	$(x)(x-5)(x^2 - 5x + 10)$		
60.20	1:2	60.21	3600	60.22	48	60.23	2/1
60.24	0	60.25	5				

1960 - 1961

61.1	5/13	61.2	3	61.3	-15	61.4	8
61.5	0	61.6	982,982	61.7	4	61.8	18
61.9	k = b	61.10	6	61.11	10	61.12	3AB
61.13	16/81	61.14	9	61.15	6	61.16	31
61.17	14,11	61.18	7	61.19	$(x^2 - 3x - 1)(x^2 + 3x - 1)$		
61.20	.77	61.21	3049	61.22	$x^3 + 9x^2 - 9x^2 - 80$		
61.23	3/16	61.24	60	61.25	32	61.26	756
61.27	17	61.28	12	61.29	0,1/2,3,7/2		
61.30	180						

1961 - 1962

62.1	9	62.2	0	62.3	6	62.4	23
62.5	42	62.6	35	62.7	31	62.8	6
62.9	$2^{2n+1}-1$	62.10	40	62.11	17/24	62.12	3/2
62.13	7.3	62.14	4/3, 3/4	62.15	18	62.16	37
62.17	3	62.18	24	62.19	3	62.20	8
62.21	67,73	62.22	(a+b)(b+c)(c+a)			62.23	1.24
62.24	2b+6c	62.25	(12,22)	62.26	4,12,26	62.27	101
62.28	34	62.29	$(1/2)(9^{1/3} + 3^{1/3} + 1)$			62.30	4

1962 - 1963

63.1	111,111	63.2	-1	63.3	3	63.4	$m^2 + n^2$
63.5	(2,3)	63.6	144	63.7	15	63.8	8
63.9	4	63.10	a+b	63.11	37	63.12	-6
63.13	3750	63.14	$-1 < x < 3$	63.15	65	63.16	305
63.17	13	63.18	625π	63.19	x=4,y=3	63.20	.24
63.21	27	63.22	.16	63.23	12	63.24	32
63.25	.32	63.26	381	63.27	4	63.28	7
63.29	a/2	63.30	(-1,0) only.				

1963 - 1964

64.1	10	64.2	12	64.3	-49/16	64.4	17
64.5	-3/8	64.6	513	64.7	70	64.8	5
64.9	5,-5	64.10	525	64.11	31	64.12	309

64.13 25,-1 64.14 130 64.15 $(7/9)^{1/4}$, $(9/10)^{1/2}$, $(4/3)^{\frac{1}{2}}$, $(3/2)^{1/3}$.
64.16 0 64.17 (3,12),(-3,-12) 64.18 21
64.19 6 64.20 98 64.21 189 64.22 65
64.23 20 64.24 (-1,1),(2,4) 64.25 3
64.26 35100 64.27 -2/3 64.28 61 64.29 4
64.30 90

1964 - 1965

65.1 54514 65.2 2,5,18 65.3 150 65.4 3
65.5 4 65.6 3 65.7 (-3,4),(7,-2),(1,-6)
65.8 2 65.9 64.1 or 64.2 65.10 none
65.11 23 65.12 60.34 65.13 43 65.14 3,-13/4
65.15 9 65.16 857142 65.17 12 65.18 20
65.19 $y^5 - 5y^3 + 5y$ 65.20 65 65.21 225
65.22 5 65.23 -32,16 65.24 1/2,-1 65.25 11
65.26 160 65.27 20 65.28 4.1 65.29 $(ay-bx)^2 +$
$(az-cx)^2 + (bz-cy)^2$ 65.30 2,4,1/2,-1/2

1965 - 1966

66.1 137,173; 139,193; 179,197 66.2 80/27 66.3 18
66.4 5 66.5 36 66.6 35 66.7 3,5,9,35
66.8 5 66.9 -5 66.10 564 66.11 5,7,8,9
66.12 1/(ab) 66.13 ±150 66.14 4 66.15 18
66.16 22 66.17 8,-1 66.18 11 66.19 3
66.20 4 66.21 119 66.22 a^2-b^2 66.23 35
66.24 (3k-1)/2 66.25 $\sqrt{5}$ 66.26 13,53 66.27 ±2
66.28 1.22 66.29 2 66.30 24

1966 - 1967

67.1 322 67.2 b+c-1, b-c+1 67.3 35.5
67.4 $(x-2)(x-1)(x^2 + 3x - 2)$ 67.5 27 67.6 36,038
67.7 $(-3+\sqrt{13})/2$ 67.8 5.9 67.9 4/(x-3) +
3/(x+2) 67.10 $\sqrt{28+8\sqrt{3}}$ 67.11 842
67.12 $2^{a+2}-1$ 67.13 3/64 67.14 30 67.15 20
67.16 27/8 67.17 ± 1.2 67.18 56/9 67.19 24
67.20 $3 < EF < 10$ 67.21 72 and 108
67.22 52 67.23 32 67.24 (4,3,9),(0,15,1)
67.25 8 67.26 21 67.27 2 67.28 19,28
67.29 4 67.30 432

1967 - 1968

68.1 1806 68.2 (2,3),(3,2),(-3,2),(-2,-3),(1/2,1/3)
(-1/2,-1/3),(1/3,1/2),(-1/3,-1/2)
68.3 4.7,1.3 68.4 25 68.5 720 68.6 (8,6)
68.7 $3 < x \leq 12$ 68.8 $(\sqrt{6} + 2)/2$ 68.9 $3y^2-(x-4)^2=0$
68.10 1008 68.11 93 68.12 (1,3,-2,-3)
68.13 .58 68.14 210 68.15 8r - 7 68.16 341

68.17 0, 2, $1+\sqrt{3}$, $1-\sqrt{3}$ 68.18 441/4   68.19 $(x+y)(x^2+xy+y^2)(5xy)$

68.20 25,17   68.21 7   68.22 (1,-1,2),(-2,2,-1),(-1,1,-2),(2,-2,1)

68.23 $0<r_2<4$ 68.24 16

68.25 3.1   68.26 8   68.27 1760^2 68.28 16

68.29 (-1,1,0)   68.30 16

1968 - 1969

69.1	39	69.2	-1,3	69.3	$6\sqrt{2} - 2\sqrt{6}$		
69.4	$28.00	69.5	325	69.6	2,223,333	69.7	bf + d
69.8	1:16	69.9	f(n)	69.10	23.1	69.11	$2(10^9+1)$
69.12	5	69.13	n-1	69.14	$4\bar{x}$	69.15	16
69.16	2025	69.17	17	69.18	$a^2/2$	69.19	307692
69.20	48	69.21	11	69.22	23,59,87	68.23	28
69.24	16	69.25	36/5	69.26	9	69.27	(3/2,1,2)
69.28	66	69.29	0,4	69.30	945		

SAMPLE SOLUTIONS, HINTS OR WHAT HAVE YOU

60.1.3.04

The question is really, "How many multiples of 196 are there between 1 and 1,000,000, which are squares?"

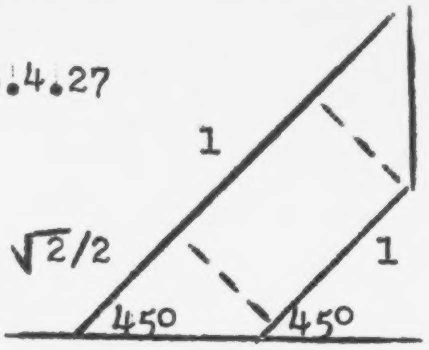
Since $(196)(5102+2/49) = 1{,}000{,}000$; and since $\sqrt{5102} = 71^+$, my answer is 71. I know that $(2)(71) = 142$, but I can't for the life of me see why 71 should be doubled! How's that for an auspicious start?

60.2.5.04

Given: $x = 1 + 1/x$
Find: $x^4 + 1/x^4$

From the given: $x - 1/x = 1$
Squaring, $x^2 - 2 + 1/x^2 = 1$
Simplify: $x^2 + 1/x^2 = 3$
Square again and simplify !

60.3.4.27



By recognition: isosceles trapezoid
Therefore, .707 + 1 + .707

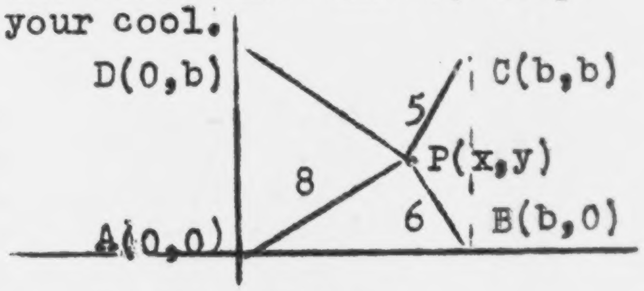
60.4.6.08

Let roots be p and q.
1.) Given: $p - q = m-1$
2.) Known: $p+q = m/2$
3.) $p \cdot q = -4$

Adding and subtracting 1.) and 2.) gives:
4.) $4p = 3m-2$
5.) $4q = -m+2$
Multiplying 4.) and 5.) yields: $16pq = -3m^2 + 8m - 4$. Use 3.), namely $pq = -4$. Solve the quadratic.

60.5.6.03

Two suggestions: Put it on the coordinate axes!; keep your cool.



Given information:
1.) $x^2 + y^2 = 64$
2.) $(x-b)^2 + y^2 = 36$
3.) $(x-b)^2 + (y-b)^2 = 25$
To be found: $(PD)^2 = x^2 + (y-b)^2$
Taking the second suggestion, we note that 1.) + 3.) - 2.) = $x^2 + (y-b)^2 = 53$. Neat, eh ?

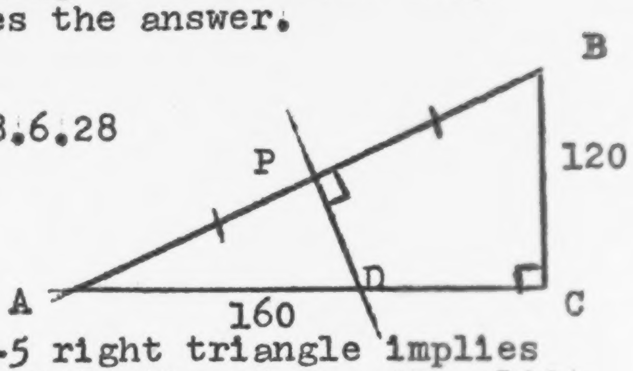
60.6.5.72

Fake it: 1.) $19\underline{16} + \underline{16} = 1932$
2.) $18\underline{66} + \underline{66} = 1932$

60.7.5.64

Key idea: work backwards.
Let b represent final number of boys. Let g represent final number of girls. Therefore, 1.) $b/g = 5/3$.
Working backwards,
2.) $(b+20)/g = 3/1$.
Solving, $b = 25$, $g = 15$.
Working back one more step gives the answer.

60.8.6.28



3-4-5 right triangle implies AB = 200; therefore, AP = 100. Since triangles ADP and ABC are similar, PD/100 = 120/160.
QED

60.9.6.20

First recognize that
$x^{2/n} - 12x^{1/n} + 27 = 0$ is equivalent to
$(x^{1/n})^2 - 12x^{1/n} + 27 = 0$.
Therefore, $x^{1/n} = 9, 3$.
And $x = 9^n, 3^n$.
Given: $9^n \cdot 3^n = 3$. Since $9 = 3^2$, $3n = 1$.

60.10.7.37

Area between the original circles is $\pi(2a)^2 - \pi a^2 = 3a^2\pi$.
Area between the "new" circles is $\pi(2a+x)^2 - \pi(a-x)^2$, which is to be set equal to $2(3a^2\pi)$. Go, man, go. No sweat.

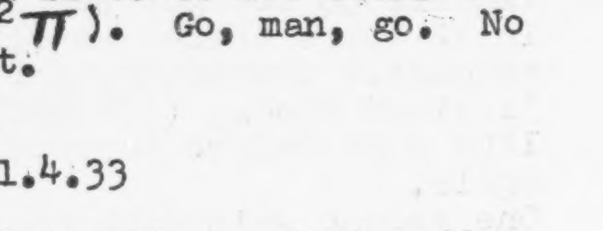
60.11.4.33

Start counting. Systematically is preferable.

60.12.6.13

Let: List Price be x
Marked Price be y.
Given: cost is .6x; profit is 1/5 y; selling price is 2/3 y. Selling Price - Cost = Profit. Therefore,
$(2/3)y - (3/5)x = (1/5)y$.
Solve carefully for y/x.

60.13.8.26



By recognition, angle CDG = 30°.
Therefore, $x = (1/2)y$.
Perimeter of AEFD = $(2)(12) + 2y + 2(1/2\ y)$.
Perimeter of EBCF = $(2)(12) + (2)(4) + 2(8-y) + 2(1/2\ y)$
Therefore $y = 6$, $x = 3$.
ABCD = $(1/2)(12+20)(4\sqrt{3}) = 64\sqrt{3}$
AEFD = $(1/2)(12+18)(3\sqrt{3}) = 45\sqrt{3}$.
Ratio in question: 64/19.

60.14.5.58

Given: $\log_{10}(x^2 - 14x + 5) = 2$.
Therefore,
$x^2 - 14x + 5 = 10^2$
Solve for x: $x^2 - 14x - 95 = 0$.

60.15.6.34

You draw the picture (can't be a regular polygon!). It is readily proveable that the sum of the vertex angles of the star is 180° (i.e., use sum of exterior angles of polygon twice and sum of the angles of a triangle five times).
Therefore, $(x-4) + (x-2) + x + (x+2) + (x+4) = 180$. And $x = 36$.

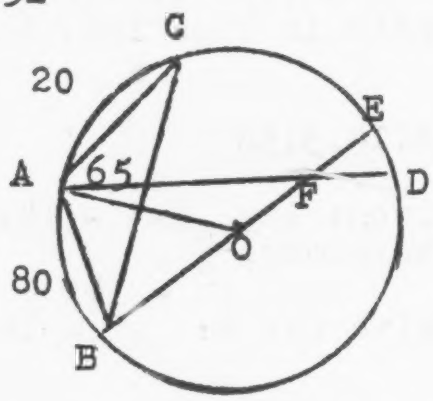
60.16.5.42

There are 10/9 centiangles per degree, 100/9 milliangles per degree. $35°20' = 106/3°$
Therefore, there are
$(106/3)(100/9) = 392\ 16/27$ milliangles; or 39c,3m.

60.17.6.29

Hint: observe.
Given: 1.) $(x-1)/(x-2) + (y-2)/(y-3) = 4$
2.) $(y-2)/(y-3) + (z-3)/(z-4) = 4$
3.) $1/(z-4) + 1/(x-2) = 2$
Look at 1.) and 2.). By substitution (using their common term), obtain $(x-1)/(x-2) = (z-3)/(z-4)$;
4.) $(z-4)(x-1) = (x-2)(z-3)$
From 3.), Solve for $(z-4)$ carefully. $z-4 = (x-2)/(2x-5)$;
Add 1 to both sides; then
$z-3 = (3x-7)/(2x-5)$.
Substituting into 4.) and multiplying through by $(2x-5)$ yields $(x-1) = (3x-7)$. Solve for x; GO!

60.18.6.51

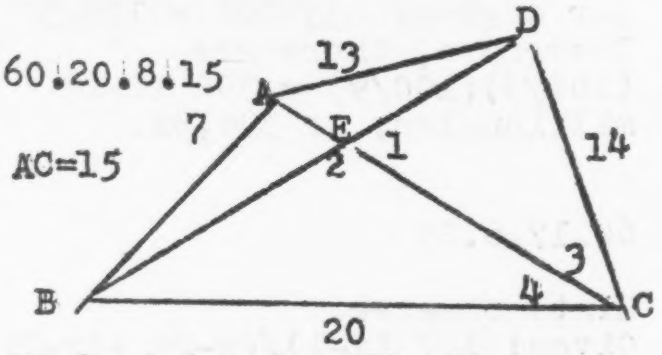


angle CAB = 130
angle CAD = 65 = angle DAB
arc AB = 80; $\overset{\frown}{ACE}$ = 100 = angle AOE. Angle AFE = 115 because arc BD = 130. Therefore, angle FAO = 15.

60.19.7.13

Multiply all the factors together and subtract the 24 obtaining $x(x^3-10x^2+35x-50)$. Use synthetic division on the second factor, remembering to make use of the rational roots theorem.

60.20.8.15



If in triangle ACD, the altitude is dropped from vertex A onto side CD, it divides the triangle up into a 9-12-15 right triangle back to back with a 5-12-13 right triangle. By knowing this, we need not use the Law of Cosines to find the cos(3); i.e., we immediately know that cos(3) = 3/5 and that sin(3) = 4/5.

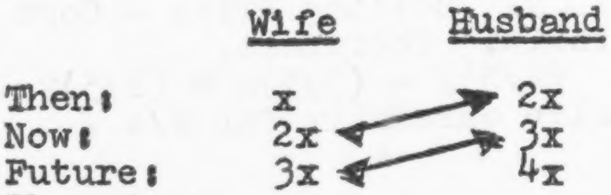
However, to find cos(4) it is necessary to use the Law of Cosines. Namely, cos(4) = (225+400-49)/600; cos(4) = 24/25; therefore sin(4) = 7/25 (i.e., 7-24-25 right triangle).

Now, note that angle 2 is the supplement of angle 1; therefore, sin(2) = sin(1). Using the Law of Sines (twice) we obtain:
(sin(4))/BE = (sin(2))/20 ;
sin(3)/ED = sin(1)/14. Since sin(2) = sin(1),
(20·sin(4))/BE = (14·sin(3))/ED.
Substituting in known values, the appropriate ratio can readily be found.

60.21.5.16

Three second loss each 2 hours implies 36 seconds per day. In 12 hours there are (12)(3600) seconds. Therefore, for the "losing" clock, (12·3600)/36 = 1200 days before correct time again.
One second gain each hour implies 24 seconds per day. Therefore, the "gaining" clock will be correct again in (12·3600)/24 = 1800 days.
Now, just find the least common multiple of 1200 and 1800.

60.22.6.37



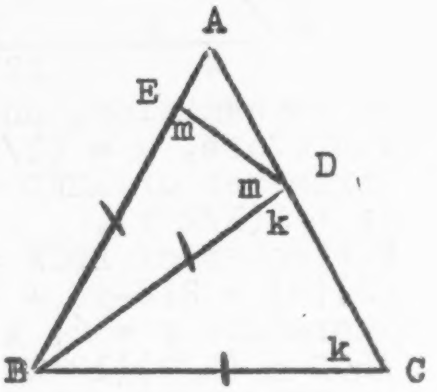
	Wife	Husband
Then:	x	2x
Now:	2x	3x
Future:	3x	4x

Therefore,

$$3x + 4x = 112 .$$

Note: the given information is denoted by the arrows.

60.23.6.26



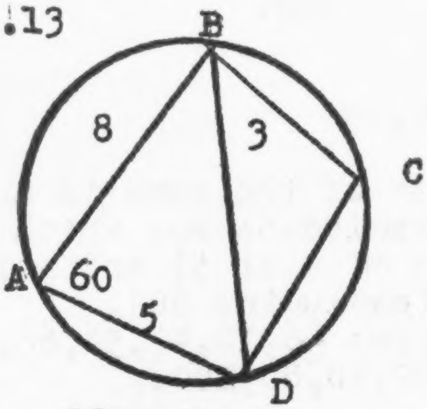
Angle EBD = 180 - 2m
angle DBC = 180 - 2k
Angle EBD + angle DBC =
angle ABC = 360 - 2(m+k) = k

Therefore, 360 = 2m + 3k, or
m = (360-3k)/2
Angle EDA = 180 - (m+k) =
180 - ((360-3k)/2 + k) = k/2
Therefore,
reciprocal of ratio in question is 1/2 .

60.24.5.41

Sum up the 3 fractions, obtaining:
$(2(b+c)(a+b) - (a+c)(a+b) - (a+c)(b+c))/((a+c)(b+c)(a+b))$.
Simplification of the numerator yields $2b^2 - a^2 - c^2 = (b^2 - a^2) - (c^2 - b^2) = 0$, since $b^2-a^2 = c^2-b^2$.

60.25.6.13



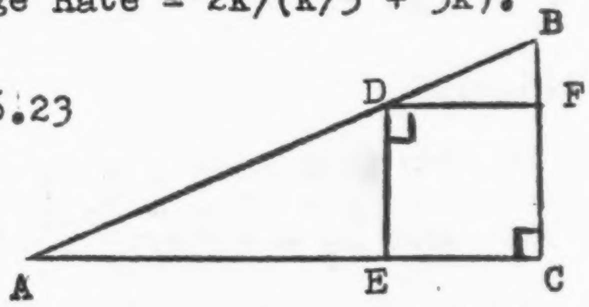
Angle C = 120; then, Law of Cosines (twice):
$(BD)^2 = 64+25-2\cdot 40\cdot 1/2 = 49$
$(BD)^2 = 49 = 9+(DC)^2-2\cdot 3(DC)(-1/2)$
Therefore,
$DC^2 + 3DC - 40 = 0.$
Solve.

1960 - 1961

61.1.5.32

Average rate ≠ average of the rates. Average rate = total distance divided by total time.
Let distance = k.
Rate 1 = 5 mph; i.e., Time 1 = k/5 hours.
Rate 2 = 1/5 mph; Time 2 = k/(1/5) = 5k hours.
Total time = k/5 + 5k hours.
Total distance = 2k.
Average Rate = 2k/(k/5 + 5k).

61.2.6.23



Obviously, triangles AED, DFB, and ACB are similar. Therefore DF/FB = AC/CB = 2/1 ; but DF = ED. Therefore, ED/FB = 2/1 = AD/DB. And finally, DB/AB = 1/3 .

61.3.6.20

Given: $\sum_{i=1}^{n} c_i = s$ and

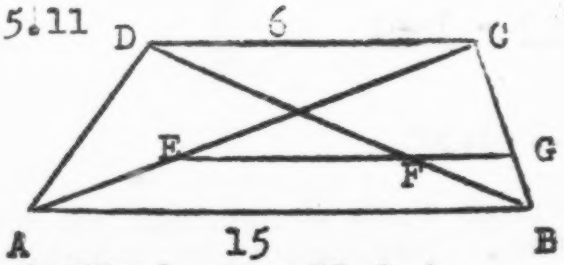
$$\sum_{i=1}^{n} (5(c_i-5)+5) = as+bn$$

But the second summation simplifies to

$$\sum_{i=1}^{n} (5c_i-20) = 5\sum_{i=1}^{n} c_i - 20n = 5s - 20\,n.$$

Therefore a+b = -15 .
For an explanation of sigma notation, consult your local Math teacher.

61.4.5.11



Segment EFG is parallel to CD is parallel to AB (Why not?). Triangles CAB and CEG are similar. Therefore,
CE/CA = 2/1 = EG/AB; EG = 10
Triangles BDC and BFG are similar. Therefore,
BF/BD = 1/2 = FG/DC; FG = 2.
Therefore, EF = 8 .

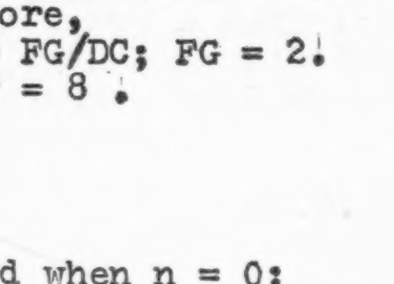
61.5.-.28

a_2 is obtained when n = 0;
Therefore $a_2 = (3a_1 - 2a_0)/3 = 8/3$. And $a_3 = (3(8/3) - 2(4))/3$

61.6.6.07

abcabc = abc(1001)
But 1001 = 7·11·13
(491)(1001) = 491,491; four primes. Only other prime which can be introduced is 2.

61.7.9.09



By recognition, triangle is a 30-60-90 right triangle.
Area of triangle = $2\sqrt{3}$
Area of Sector I = $(\pi/12)(4+2\sqrt{3})$
Area of Sector II = $(\pi/4)(4-2\sqrt{3})$
Area of Sector III = $(\pi/6)(12-6\sqrt{3})$

Area bounded by circles = Area of triangle minus the sum of the areas of the three sectors. Go to it. Carefully.

61.8.4.92

3(x+3)-3(x-3) = x . Solve.

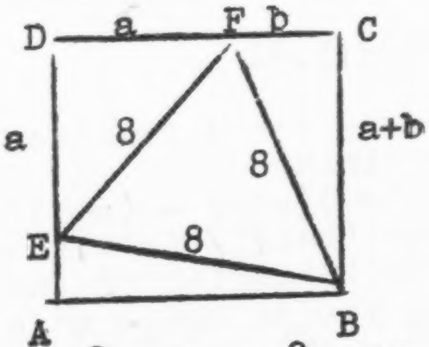
61.9.7.37

a^b maps into $(a^2)^{b^2} = a^{2b^2}$

$((a^b)(a^b))^k = (a^{2b})^k = a^{2bk}$

Therefore, $2bk = 2b^2$.

61.10.7.10



$a^2 + a^2 = 64$; $b^2 + (a+b)^2 = 64$

$a^2 = 32$, $a = 4\sqrt{2}$; $b = -2\sqrt{2} + 2\sqrt{6}$. QED.

61.11.6.09

The name of the game is multiples of 6 (excluding any which have factors of 4 or 5) and multiples of 20 (excluding 60).
Therefore: {6,18,42,54,66,78} and {20,40,80,100.}

61.12.6.21

$1/x^3 + 1/y^3 = (x^3+y^3)/(xy)^3 =$

$((x+y)(x^2-xy+y^2))/(xy)^3 =$

$((x+y)(x^2+2xy+y^2-3xy))/(xy)^3 =$

$((x+y)((x+y)^2 - 3xy))/(xy)^3 =$

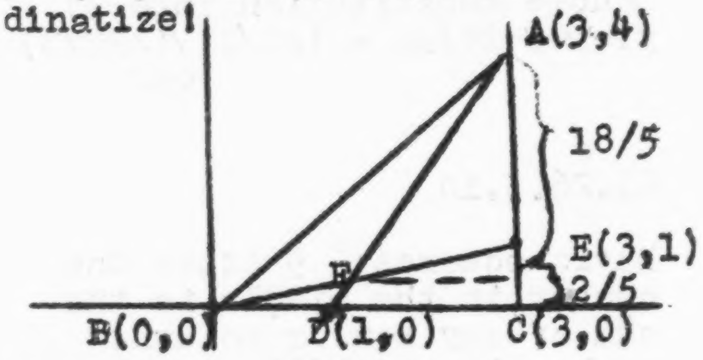
$(A(A^2-3B))/B^3$.

61.13.6.43

After Step 1: 2.3 of container is water. After Step 2: (2/3)(2/3) of container is water. Two more times.

61.14.6.07

Let the constant of proportionality be equal to 1 in each case (Why not ?). Let AC be perpendicular to BC (Why not ?). Then coordinatize!



Line BE: $y = (1/3)x$
Line AD: $y = 2x-2$
Therefore, F(6/5,2/5).
By similar triangles,
$DF/FA = (2/5)/(18/5)$

61.15.5.35

$(\log_k 125)(\log_{25} k)(\log_k k^4) =$

$(\log_5 5^3/\log_5 k)(\log_5 k/\log_5 5^2)(4)$

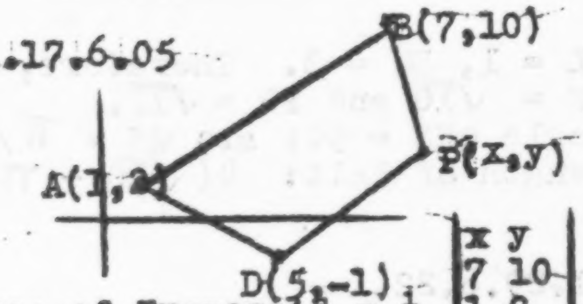
$(3)(1/2)(4) = 6.$

61.16.5.20

Key: Work backwards.

$$\begin{array}{lcl} 32 & \leftarrow & 0 \\ 48 & \rightleftarrows & 16 \\ 56 & \rightleftarrows & 24 \\ 60 & \rightleftarrows & 28 \\ 62 & \rightleftarrows & 30 \\ & \searrow & 31\text{, answer.} \end{array}$$

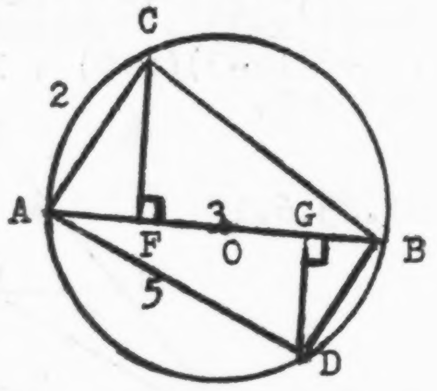
61.17.6.05



Area of Trapezoid $= \frac{1}{2}\begin{vmatrix} x & y \\ 7 & 10 \\ 1 & 2 \\ 5 & -1 \\ x & y \end{vmatrix} =$

$\frac{1}{2}((10x+14-1+5y)-(7y+10+10-x)) =$
$125/2.$ Or,
1) $11x - 2y = 132$
From slopes: $(y+1)/(x-5) = 4/3$
2) $4x - 3y = 23.$
Solve 1) and 2) simultaneously for the values in question.

61.18.6.17



Using Mean Proportional Theorem:
$2^2 = (AF)(AB)$
$5^2 = (AG)(AB) = (AF+3)(AB)$,
since $AG = AF+3$.
Therefore, $4/AF = 25/(AF+3)$;
First find AF, then AB.

61.19.5.05

$x^4-11x^2+1 = x^4-2x^2+1 - 9x^2 =$

$(x^2-1)^2 - (3x)^2 = (x^2-3x-1)(x^2+3x-1)$

61.20.5.09

$$\frac{\sqrt{8+2\sqrt{15}}-\sqrt{8-2\sqrt{15}}}{\sqrt{8+2\sqrt{15}}+\sqrt{8-2\sqrt{15}}}$$ becomes

when rationalized by 1 in the form of

$\sqrt{8+2\sqrt{15}}+\sqrt{8-2\sqrt{15}}$ divided by itself, the simpler expression $4\sqrt{15}/(16+2\sqrt{64-60}) = \sqrt{15}/5 = \sqrt{3}/\sqrt{5}$. $\sqrt{3} \approx 1.732$, $\sqrt{5} \approx 2.236$. Go to it.

61.21.5.09

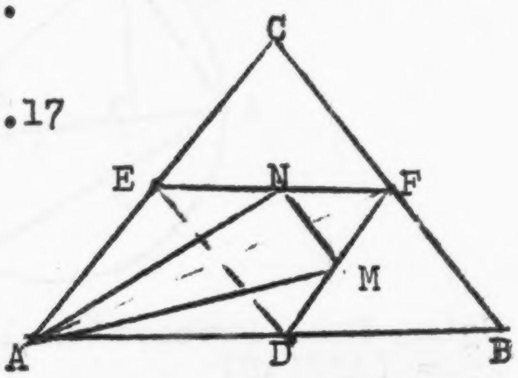
Divisibility by 11 rule: $(2+8+1)-(y+3+x)$ must be divisible by 11. $11-(x+y+3) = 11q$.
Therefore, $x+y = 8$,~~19~~.

Divisibility by 8 rule:
31x must be divisible by 8.
Therefore $x = 2$ and $y = 6$.
$88z = 268312$. Divide.

61.22.5.25

$(x+3)(x-3)$; $(x+9)(x-3)$;
$(x+9)(x+3)$. Therefore
$((x+3)(x-3)(x+9) + 1$ does
the job.

61.23.6.17



1) $BDF = CEF = AEF = (1/4)ABC$
Therefore, $FED = (1/4)ABC$
$FNM = (1/4)FED = (1/16)ABC$.

2) $AEN = (1/2)AEF$; $ADM = \frac{1}{2}ADF$
Therefore,

$$\begin{aligned} AEN + ADM &= (1/2)(EAF + ADF) \\ &= (1/2)\ AEFD \\ &= (1/2)((1/2)ABC) \\ &= (1/4)ABC \end{aligned}$$

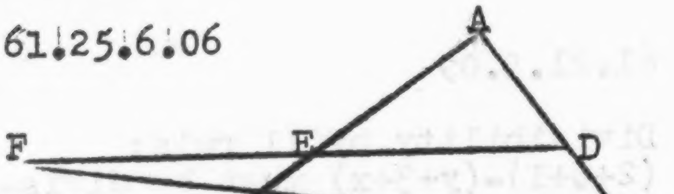
3) $FNM + (AEN+ADM) = (5/16)ABC$

$$\begin{aligned} AMN &= ADFE - (5/16)ABC \\ &= (1/2)ABC - (5/16)ABC \\ &= (3/16)\ ABC \end{aligned}$$

61.24.5.11

If base population is taken
as 1, then the
Hourly Gain is: 1x
New Population is: 1+x
Hourly loss is: $(1+x)x$
Net Loss = .36 = Hourly Loss -
Hourly Gain = $x^2 + x - x$.
Solve.

61.25.6.06



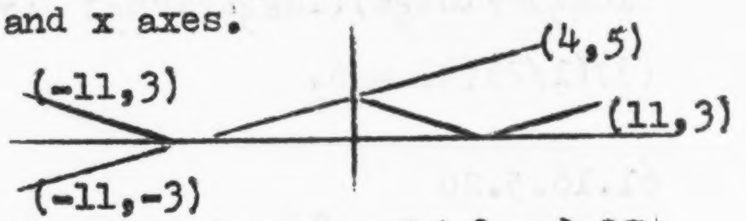
Given: $ED/FE = 1/2$
1) Triangles BCG and EFG are similar.
2) $2/3 = BC/FE$; $BC = (2/3)FE$
3) Triangles AED and ABC are similar. Therefore,
$24/AB = ED/BC$.
Double substitution into 3)
yields $24/AB = (FE/2)/(2FE/3)$.
QED

61.26.5.10

Magic squares: 9 times the number in the middle is the sum of any row or column. Number in themiddle is the middle number of the 81 term sequence; namely,
$4 + (41-1)(2) = 84$; $9 \cdot 84 = 756$
Obviously.

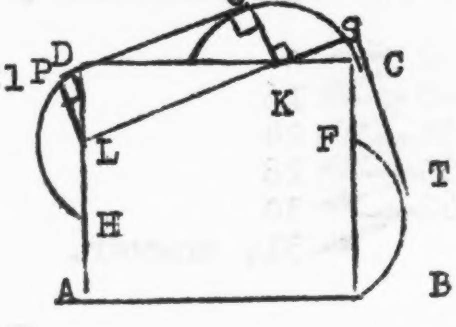
61.27.6.05

Key idea: 1. Angle of incidence equals angle of reflection.
2. Reflect in the y and x axes.



Distance formula: $15,8 \Longrightarrow 17$.

61.28.-.21



$DL = 1$, $DK = 3$. Therefore,
$LK = \sqrt{10}$ and $PQ = \sqrt{10}$.
Angle $QKS = 90$; arc $QS = \pi/2$.
Length of Belt: $4(\sqrt{10} + \pi/2)$.

61.29.6.22

$y^2 - 3.5y + 2.5 = 0$
$2y^2 - 7y + 5 = 0$; $y = 5/2, 1$
Therefore, $x^2-3.5x+2.5=2.5,1$.
Go.

61.30.5.25

Boy #1: Rate is $4k$
Boy #2: Rate is $5k$
Since time is the same for both,
$(x+y+10)/5k = (y+x-10)/4k$
And $x+y = 90$
$2(x+y) = \text{perimeter} = 180$.

1961 - 1962

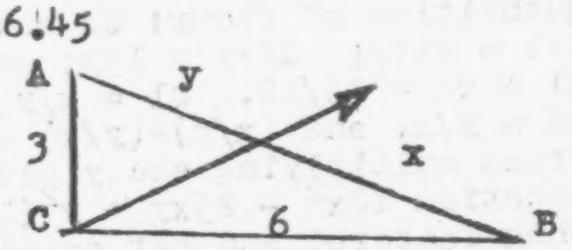
62.1.5.36

$DDD = D(111) = D(3)(37)$
Therefore, $I = 3$, 6 or 9.
Since $(I)(DA) = DDD$; there are 3 choices.
$(3)(37) = 111$ NG
$(6)(37) = 222$ NG
$(9)(37) = \underline{333}$ OK

62.2.5.11

Given: $(a + 1/a)^2 = 3$; or
$a + 1/a = \sqrt{3}$.
$a^2+2+1/a^2 = 3$; $a^2+1/a^2 = 1$;
$a^2 -1 + 1/a^2 = 0$
Therefore $a^3 + 1/a^3 =$
$(a+1/a)(a^2-1+1/a^2) = \sqrt{3}\cdot 0 = 0$.

62.3.6.45



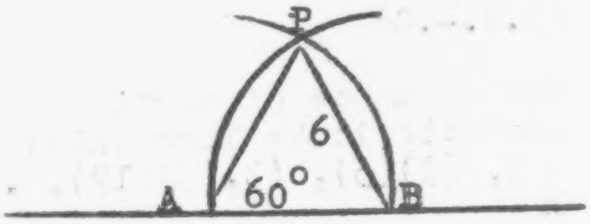
$(x+y)^2 = 3^2+6^2 = 45$
But an angle bisector theorem says: $x/6 = y/3$; $x = 2y$.
Therefore, $(3y)^2 = 45$; $y = \sqrt{5}$
$x = 2\sqrt{5}$.
Area $= (1/2)ax\sin(B) =$
$(1/2)(6)(2\sqrt{5})(3/\sqrt{45})$

62.4.6.33

$3x-4 = (k(y+5))/(z-5)^2$
or, $k = ((3x-4)(z-5)^2)/(y+5)$
Using given information, $k = 13$.
Therefore, $3x-4 = (13)(5)$.

62.5.7.06

Remember: TOP and BOTTOM !!!



$AP \le 6$, $BP \le 6$
Sector $PAB = 6\pi$
Lune $AP = 6\pi - (9\sqrt{3})$
Area of upper part $= 6\pi + 6\pi - 9\sqrt{3}$.
Total area $= 24\pi - 18\sqrt{3}$.

62.6.4.37

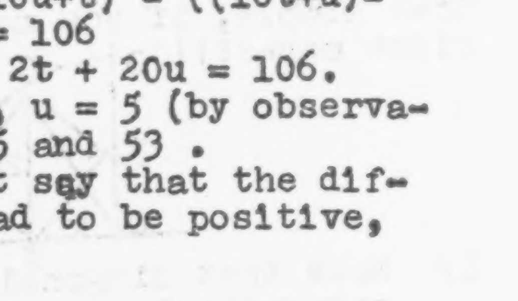
$(10t+u + 10u+t) - ((10t+u)-(10u+t)) = 106$
Therefore $2t + 20u = 106$.
Try $t = 3$, $u = 5$ (by observation). 35 and 53.
(It didn't say that the difference had to be positive, did it ?)

62.7.-.03

Since $1-x^2 = (1+x)(1-x)$ we should observe that the substitution $A = (1+x)^{1/5}$, $B = (1-x)^{1/5}$ reduces the problem to
$$A^2 + 2B^2 = 3AB;\ A = \{2B, B\}$$
$A = 2B$ yields the solution $x = 31/33$; $A = B$ yields $x = 0$.

62.8.6.28

$ABCD = 144$



Triangles AEG and CGB are similar. Therefore, $AEG/CGB = 1/9$,

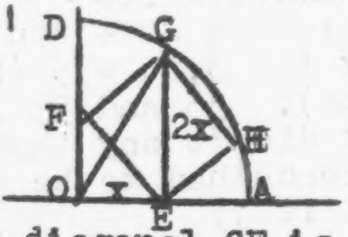
since AE/BC = 1/3 .
1) AEG + AGB = (1/3)(1/2)(144)
i.e., AEB = 1/3 ABC
2) AGB + GBC = AGB + 9(AEG) = 72
From 1), (24 - AEG) + 9AEG = 72
Therefore AEG = 6.

62.9.-.06

The sequence of partial products looks like this:
{3, (3)(5), (3)(5)(17), ...}
Or
{3, 15, (16-1)(16+1), ...}
Or
{3, 15, 255, 256^2-1, ... }
Or
{2^2-1, 2^4-1, 2^8-1, $2^{16}-1$, ...}
Therefore, $2^{2\cdot 2^n} - 1$.

62.10.7.04

Très simple, if you make two right moves!!!



1. Note that diagonal GE is perpendicular to OA.
2. Draw radius OG.
3. OE = x
4. FE = $\sqrt{2}x$
5. GE = $\sqrt{2}(\sqrt{2}x) = 2x$

Therefore, $x^2 + (2x)^2 = 100$;
$x^2 = 20$. $FE^2 = (\sqrt{2}x)^2 = 2x^2$.
QED.

62.11.-.52

For speed, please observe the following:
(N1/D1)/(N2/D2)
D1: (38 + (63/99))-(38+(27/99)) = 36/99
Note: 38 and (63/99) means 38 + (63/99).
D2: 81/99 -(5/11)(9/9) = 36/99
N2: (62+3/9)-(60+1/3) = 2
Therefore, fraction reduces to
((2/3+3/4)/2)(12/12) = (8+9)/24 .

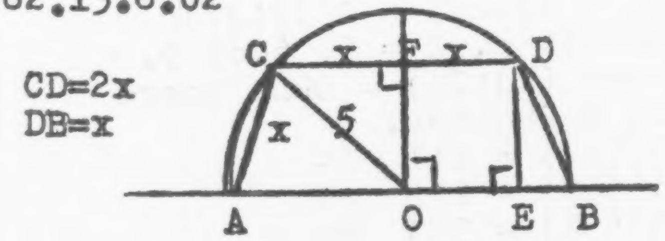
62.12.-.63

$4^{2x} + 2^{2x+1} = 80$

$(2^2)^{2x} + 2^{2x}\cdot 2^1 = 80$

$(2^{2x})^2 + 2\cdot 2^{2x} - 80 = 0$
A quadratic in 2^{2x} .
Therefore, factoring and solving gives
$2^{2x} = \{-10, 8\}$
2x = ~~$\log_2(-10)$~~, $\log_2(8)$
And
2x = 3.

62.13.6.02

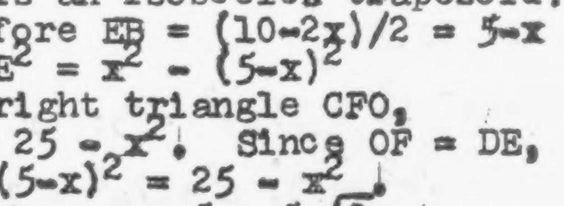


ABDC is an isosceles trapezoid!
Therefore EB = (10-2x)/2 = 5-x
And $DE^2 = x^2 - (5-x)^2$
From right triangle CFO,
$OF^2 = 25 - x^2$. Since OF = DE,
$x^2 - (5-x)^2 = 25 - x^2$.
Solving, $x = -5 + 5\sqrt{3}$.

62.14.7.25

Quantities of flour: Q1 and Q2
x·1 = Q1·y; Q2·x = 1·y; and
Q1 + Q2 = 25/12. Q1 = x/y,
Q2 = y/x, and (x/y)+(y/x) = 25/12
Cross multiplying etc yields the equation $12x^2 - 25xy + y^2(12) = 0$
When factored and set equal to zero, the ratios x/y = {4/3, 3/4} should appear.

62.15.5.10



III = $(9\pi x^2)/8$; II = $(4\pi x^2)/8$
I = $(\pi x^2)/8$; Area of region = $(4\pi x^2)/8$. Sum of circumferences equals $3\pi x$. QED.

62.16.5.35

Good ole 111 again.
uuu = u(111) = u(3)(37)
(3)(7) = 21; (21)(3$\underline{7}$) = 77$\underline{7}$.

62.17.-.53

$(2^{n+2}-2^n+3)/(2^n+1) =$

$((2^2)(2^n)-2^n+3)/(2^n+1) =$

$(2^n(4-1)+3)/(2^n+1) =$

$(3(2^n+1))/(2^n+1) = 3$
Or simply substitute in a couple values for n and watch the 3 pop out.

62.18.5.65

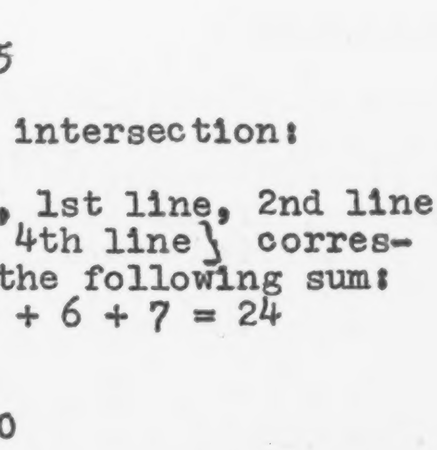
Points of intersection:

{2 circles, 1st line, 2nd line, 3rd line. 4th line} corresponds to the following sum:
2 + 4 + 5 + 6 + 7 = 24

62.19.-.20

$(x+1)^2 > 5x-1$; $(x+1)^2 < 7x-3$
$x^2-3x+2 > 0$ $\quad x^2-5x+4 < 0$
1) $(x-2)(x-1) > 0$
2) $(x-4)(x-1) < 0$
Truth set for 1) $x < 1$ or $x > 2$
Truth set for 2) $1 < x < 4$
Only integer in the intersection of two sets is 3.

62.20.6.26



Law of Cosines: (angle D = 120)
$s^2 = 9+25-(2)(15)(-1/2) = 49$
More Law of Cosines:
1) $AD^2 = 25+49-2(35)\cos(C)$
2) $AD^2 = 9+49-2(21)\cos(B)$

Since cos(B) = -cos(C), 2) becomes
$$AD^2 = 58 + 42\cos(C)$$
Therefore
$$58 + 42\cos(C) = 74 - 70\cos C$$
cos(C) = 1/7. Solve for AD using 1) or 2).

62.21.5.24

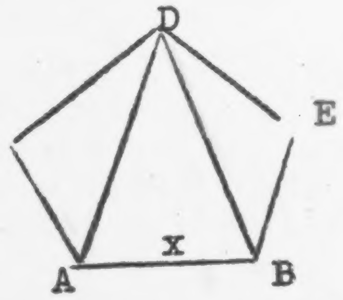
$4891 = 4900 - 9 = (70)^2 - (3)^2 = (70+3)(70-3)$.

62.22.-.19

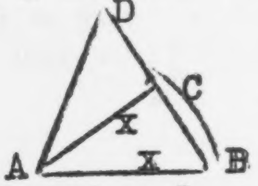
You can spot the lurking factors by first doing some partial factoring, thusly:
$(a(a+b)+c(a+b))(b(b+c)+a(b+c))\cdot(c(a+c)+b(a+c)) = (a+b)(a+c)\cdot(b+c)(b+a)(a+c)(c+b)$.

QED

62.23.8.03



Since triangle DEB is isosceles and simce angle E = 108, angle ABD = 72 = angle DAB; angle ADB = 36. Look at triangle ABD; swing an arc of radius x centered at A intersecting side BD at C.

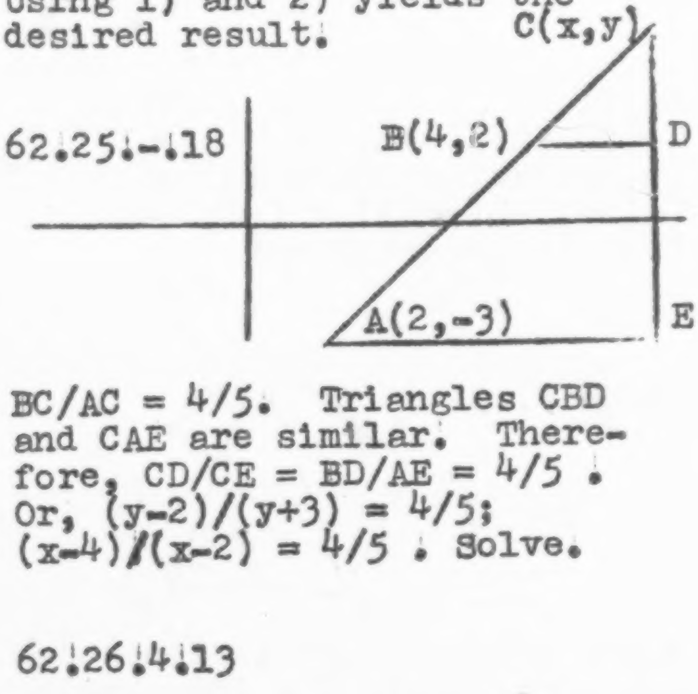


Note that both triangles ABC (by design) and ACD are isosceles. Therefore AC = DC = x; BC = 2-x. Also triangles ABC and ABD are similar. Therefore DA/AB = AB/BC; i.e., 2/x = x/(2-x). Solving, $x = -1+\sqrt{5}$.
Note: This problem must be done by some sort of similar triangles unless you happen to know the cosine or sine of 72 or 36 degrees, the finding of which has essentially been done above.

62.24.-.004

Take the $(4x^2+9y^2+z^2)(a^2+b^2+c^2) = (2ax+3by+cz)^2$; expand both sides and put all terms on the left; you should then recognize the following: $(3ay-2bx)^2+(az-2cx)^2+(3cy-bz)^2 = 0$. But this will only be true if 1) $3ay-2bx = 0$; which is to say, $y = (2bx/3a)$. And 2) $az-2cx = 0$; which is to say $z = (2cx)/a = (6cx)/3a$. (All that's necessary!) Since $rx = 3az$ and $qx = 3ay$, $r+q = (3a(y+z))/x$. Using 1) and 2) yields the desired result.

62.25.-.18



BC/AC = 4/5. Triangles CBD and CAE are similar. Therefore, CD/CE = BD/AE = 4/5. Or, $(y-2)/(y+3) = 4/5$; $(x-4)/(x-2) = 4/5$. Solve.

62.26.4.13

$(x+y)/2 + z = \{23,27,34\}$
1) $x+y+2z = 68$
2) $x+z+2y = 54$
3) $y+z+2x = 46$
2) - 3) : $y-x = 8$
1) - 2(3)): $-3x-y = -24$
$x = 4$. Go.

62.27.7.28

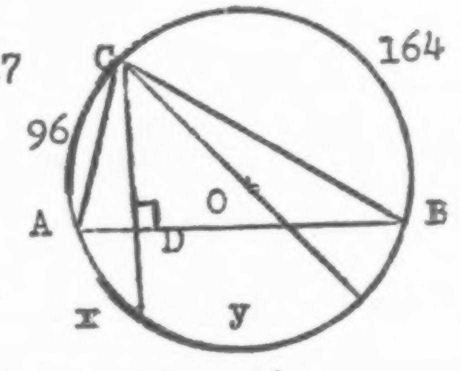
$3^{48}-1$: $3^2 = 9 \equiv 2$, mod 7

$2^3 = 8$; therefore, $(3^2)^3 = 3^6$ is congruent to 1 mod 7. And $(3^6)^8 \equiv 1$, mod 7. Therefore $3^{48}-1$ is divisible by 7.

$3^5 = 243 \equiv 1$, mod 11.
$(3^5)^9\cdot 3^4 \not\equiv 1$, mod 11; therefore $3^{48}-1$ is not divisible by 11.

$3^4 = 81 \equiv 8$, mod 73;
$3^6 = 3^2\cdot 3^4 \equiv 9\cdot 8 \equiv -1$, mod 73
$(3^6)^8 = 3^{48} \equiv (-1)^8 \equiv 1$, mod 73
Therefore $3^{48}-1$ is divisible by 73.

62.28.6.17

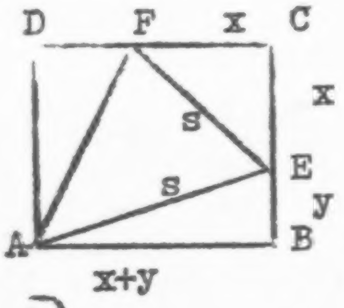


$96+x+y = 180$; $x+y = 84$
$(x+164)/2 = 90$; $x = 16$, $y = 68$
and angle DCO = 34.

62.29.-.16

Since $x^3-y^3 = (x-y)(x^2+xy+y^2)$, the rationalizer will be
1) $((3^{1/3})^2 + (1)(3^{1/3}) + 1)$.
That is, if the given expression is multiplied numerator and denominator by 1), the result should be the required answer.

62.30.7.18



$$\left.\begin{array}{l}1)\ s^2 = 2x^2 \\ 2)\ (x+y)^2+y^2 = s^2\end{array}\right\} x^2=2(xy+y^2)$$

3) Sum of areas of 4 triangles equals area of square.
$(2)(1/2)(x+y)(y) + (1/2)x^2 + (2x^2\sqrt{3})/4 = 8 + 4\sqrt{3}$.
$(xy+y^2) + x^2((1+\sqrt{3})/2) = 8+4\sqrt{3}$
Substituting from 1) and 2),
$(x^2/2)+x^2((1+\sqrt{3})/2) = 8+4\sqrt{3}$.
Therefore $x^2 = 8$, $2x^2 = 16$.
QED

63.1.5.25

aaa,aaa = a(111,111) =
a(111)(1001) = a(3)(37)·
(7)(11)(13). Therefore a = 1.

63.2.6.29

Given: $u_2 = 10$, $u_5 = 33$.

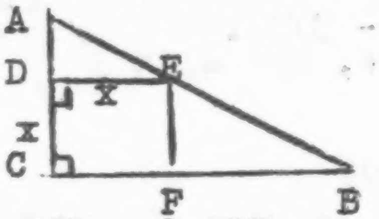
$u_3 = 2(u_2) + a = 10 + a$

$u_4 = 2(10+a)+a = 20 + 3a$

$u_5 = 2(20+3a)+a = 40+7a = 33$

63.3.6.49

Note: an appeal for the answer 5 was granted for this one --- but I haven't been able to figure out why.

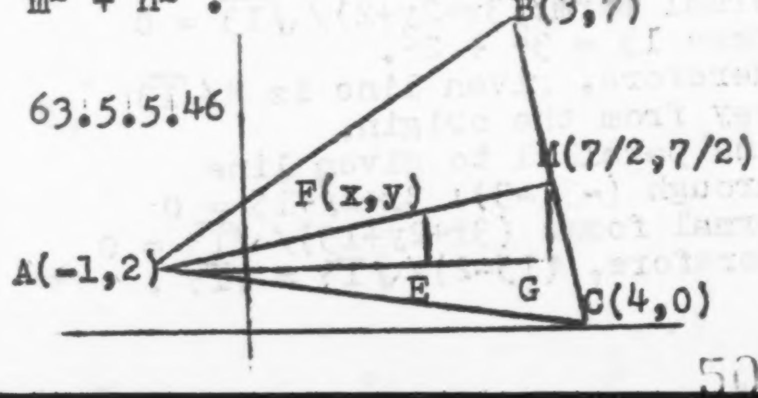


Triangles ACB and ADE are similar; therefore AD/x = AC/BC. But AD = AC - x; and x = (2/5)AC. Therefore AD = AC - (2/5)AC = (3/5)AC. By substitution, (3/5)AC/(2/5)· AC = m/2 ; m = 3.

63.4.6.25

1) $ax + by = m$
2) $bx - ay = n$
3) $a^2 + b^2 = 1$
Square 1): $(ax)^2 + 2abxy + (by)^2 = m^2$.
Square 2): $(bx)^2 - 2abxy + (ay)^2 = n^2$. Add: $x^2 + y^2 = m^2 + n^2$.

63.5.5.46



Triangles AEF and AGM are similar and AF/AM = 2/3.
Therefore AE/AG = 2/3 = (x+1)/9/2
And x = 2.
FE/GM = 2/3 = (y-2)/((7/2)-2);
And y = 3.

63.6.4.09

815 - 383 = 432 = (3)(144)
815 - 527 = 288 = (2)(144)
527 - 383 = 144
Therefore (???), 144.
Kick it around --- maybe you'll discover why.

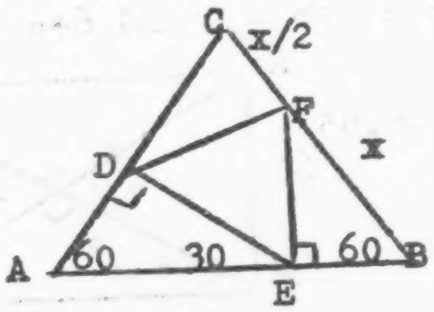
63.7.6.05

Exasperation! Perspiration, but no inspiration. Want a sample solution? Send a self-addressed, stamped envelope to:

Mr. George Lenchner
% Valley Stream North H.S.
Franklin Square, New York

They tell me he's a very smart man. Anyhow, he started this whole league so let him answer this %$#)*&"+_&$'#%" question.

63.8.6.46



Let FB = x = AE = DC.
Therefore EB = x/2 = AD.
x + x/2 = 24 QED

63.9.5.48

$(x/2 + 10)/2 + 10 = x^2$.
Solve it.

63.10.5.54

I have beat this one to death ---- 5 similar triangles, proportions by addition; extended lines and mean proportionals. to the hypotenuse; equivalent areas. Nowheresville. However: apparently the relationship between AG and the magnitudes a and b does not depend upon the position of Q; i.e., it is an invariant relationship. Therefore, let Q approach C, in which case CG approaches 0, AG approaches AC and E approaches F; therefore AG approaches AC approaches BD approaches a+b.

63.11.5.42

More ttt = t(111) = t(3)(37)
Finagle! t = 3.

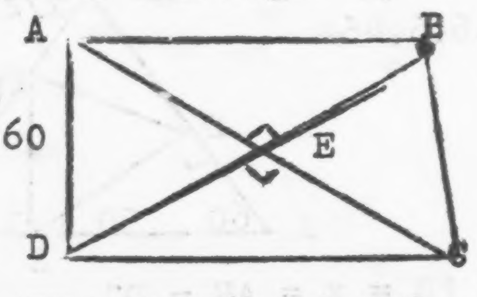
63.12.7.12

$x^3+ax+b=0$; $x = \sqrt[3]{4} + \sqrt[3]{2}$

$x^3 = (\sqrt[3]{4} + \sqrt[3]{2})^3 =$

$4 + (3)(2)(\sqrt[3]{4}) + 3(2)(\sqrt[3]{2}) + 2$
$= 6 + 6(\sqrt[3]{4}+\sqrt[3]{2})$.
Therefore, $6+6(\sqrt[3]{4}+\sqrt[3]{2}) + a(\sqrt[3]{4}+\sqrt[3]{2}) + b = 0$. The only way for the LHS to be zero is for: $6+b = 0$ and $6+a = 0$.

63.13.6.05

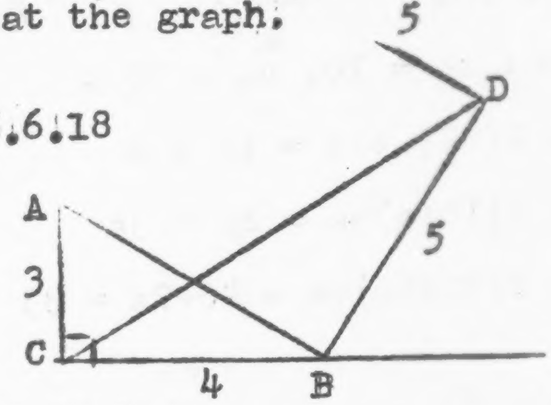


Triangles AED and BEA and ADC are similar. AD/DC = 3/4; 60/DC = 3/4; DC = 80
They are all 3-4-5 right triangles; only the factors are missing.
AE/ED = 3/4 sez that AE = 36, DE = 48. i.e., AD = (5)(12) = 60.
Since EB/AE = 3/4, EB = 27.
Therefore AB = (9)(5) = 45.
You find the area.

63.14.6.64

Parabola: $y = x^2-x-2$
Line: $y = x + 1$
For points of intersection: $x^2 -x -2 = x + 1$; $x = \{-1,3\}$
Therefore, $-1<x<3$. Just look at the graph.

63.15.6.18



Law of Cosines:
$CD^2 = 16+25-(2)(4)(5)\cos(CBD)$
$= 41 - 40 \cos(CBD)$
$\cos(CBD) = \cos(CBA + ABD) =$
$\cos(CBA)\cos(ABD)-\sin(CBA)\cdot\sin(ABD)$
$= 0 - 3/5$.
Therefore, $CD^2 = 65$.

63.16.5.03

Quick Counting Problem!

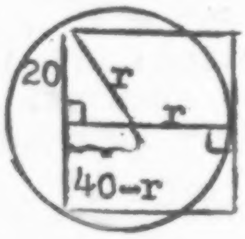
1 origin: 1
9 on each axis: 36
67 in each quadrant: 268
Total: 305
Quadrant breakdown:
67 = 4(9) + 1(8) +2(7)+5+4.
i.e., (1,1-9) (5,1-8)
(2,1-9) (6,1-7)
(3,1-9) (7,1-7)
(4,1-9) (8,1-5)
(9,1-4)

63.17.6.49

Given: $3x+2y+2 = 0$
Normal form: $(3x+2y+2)/\sqrt{13} = 0$
Where $13 = 3^2 + 2^2$.
Therefore, given line is $2/\sqrt{13}$ away from the origin.
Line parallel to given line through (-3,-3): $3x+2y+15 = 0$
Normal form: $(3x+2y+15)/\sqrt{13} = 0$
Therefore, $(15-2)/\sqrt{13} = \sqrt{13}$.

63.18.6.32

More 3-4-5 (Camouflaged!)



$20^2 + (40-r)^2 = r^2$
$80r = 2000$; $r = 25$ QED

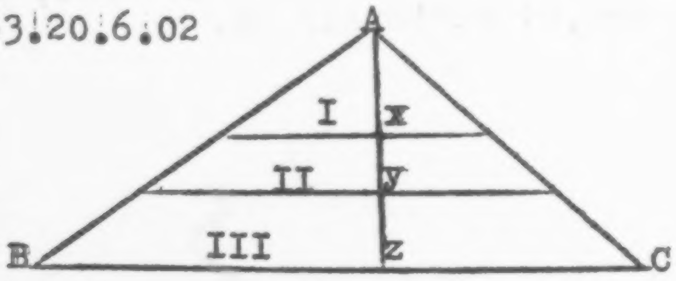
63.19.6.77

$3(2^x) = 16y$ $3^x = 27y$
$3(2^x) = 2^4y$ $3^x = 3^3y$
$3(2^{x-4}) = y$ $3^{x-3} = y$
Therefore,

$$3(2^{x-4}) = 3^{x-3}$$
$$(2^{x-4}) = 3^{x-4}$$

Only true if $x-4 = 0$.
Therefore $x = 4$, $y = 3$.

63.20.6.02



$x=x$; $y = ka$; $x+y+z = a$
Ratio of $I/(I+II) = 1/2$
Therefore, 1) $x/(x+y) = \sqrt{2}/2 = x/(x+ka)$
Ratio of $(I+II)/(I+II+III) = 2/3$. Therefore, 2) $(x+y)/(x+y+z) = \sqrt{2/3} = (x+ka)/a$.
From 1), $x = (\sqrt{2}ka)/(2-\sqrt{2}) = ka(\sqrt{2}+1)$
Substituting into 2),
$(ka(1+\sqrt{2})+ka)/a = \sqrt{2/3}$.
Therefore, $k = (\sqrt{2} - 1)/\sqrt{3}$.
Go.

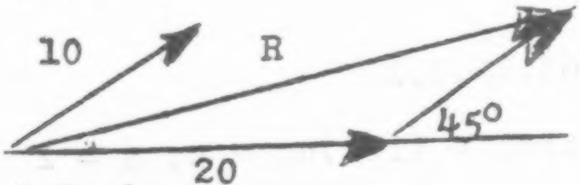
63.21.5.72

Since $(13)(769 + (3/13)) = 10{,}000$; and since $(27)^2 = 729$, $(28)^2 = 784$, the answer is obvious.

63.22.5.61

$2\pi r_1 - 2\pi r_2 = a$
$2\pi(r_1 - r_2) = a$
$r_1 - r_2 = ka$
Substitute: $2\pi(ka) = a$.
Solve. Try $\pi = 3.14159$.

63.23.6.24

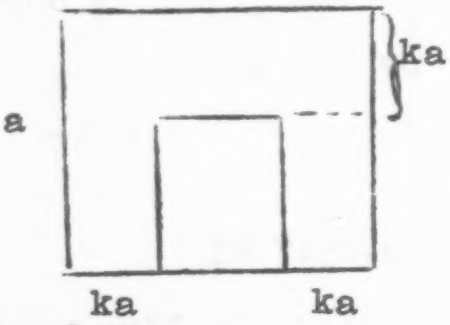


Law of Cosines:
$R^2 = 400 + 100 - 2(10)(20)\cos(135)$
$= 500 - 400(-\sqrt{2}/2)$.
Go.

63.24.7.29

This problem is just too much. The A'B'C' triangle does not exist at all since you don't find too many 12-24-36 triangles around any more. Therefore, merely find the area of triangle ABC. Area = $(1/2)(8)(12)\sin(x)$, where x is the angle formed by the sides measuring 8 and 12. Since the altitude of an isosceles triangle bisects the base, by good ole Albert Pythagorus we know the altitude is $8\sqrt{2}$. Therefore $\sin(x) = (2\sqrt{2})/3$. QED.

63.25.6.14



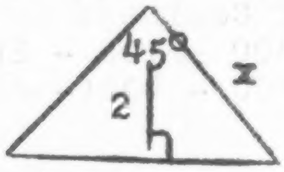
Area of rectangle:
$(a-2ka)(a-ka) = (1/4)a^2$
Or $(1-2k)(1-k) = 1/4$
$k = (3 \pm \sqrt{3})/4$; reject the +.
Arithmetize it.

63.26.4.46

$x^2 + x = 145,542$
$(x)(x+1) = 145,542$
Product of two consecutive integers! Fake it (almost a perfect square.)!
$(370)^2 = 136,900$
$(380)^2 = 144,400$ Success!
Obviously (381)(382) .

63.27.7.17

Area = $(1/2)ap = p$; $a = 2$



$\cos(45^\circ/2) = 2/x$;
$x = 2/(\sqrt{(1+\cos(45^\circ)/2)})$
$x = 2\sqrt{4 - 2\sqrt{2}}$

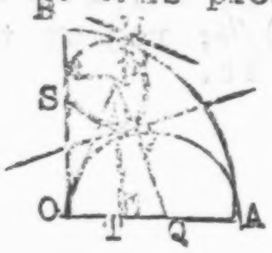
63.28.7.60

Practice makes perfect. If at first you don't succeed, try and try again. With a little bit of luck. There are only 10 digits to work with, so after a finite amount of time the solution is bound to appear. One "out of the blue" hint, however: C-E = -3; C*A = E; A = -3. If that last statement is read as A equals the additive inverse of 3, base 10, then A obviously equals 7.

By the way:
$$\begin{array}{r} 7743 \\ \underline{4377} \\ 3366 \end{array}$$

63.29.7.41

Here's one I've beat to death with no success in the past; but look at this procedure.



OA = 2a, OQ = a, SP = r = OT;
TQ = a-r, segment PVQ = a+r.
Using right triangle PTQ,
$PT^2 = (a+r)^2 - (a-r)^2$; $PT^2 = 4ar$
$OP^2 = OT^2 + PT^2 = r^2 + 4ar$
Major assumption: OPU is a straight line segment. (Why not?)
Therefore, OP + PU = 2a
Therefore,
$$\sqrt{r^2+4ar} + r = 2a$$
$$r = a/2$$
Alternate procedure (includes proof that OPU is a line segment): Try coordinate geometry with O(0,0).

63.30.5.44

$y = x^2 - 2x - 3$
$(y+4)/(x-1) = -2$; $y = -2x-2$
Therefore, $x^2-2x-3 = -2x-2$
$x = \{1,-1\}$
However!
(1,-4) is not on the given "line", which is really a line with a hole in it! Trickee, eh?
(-1,0) obviously OK.

1963 - 1964

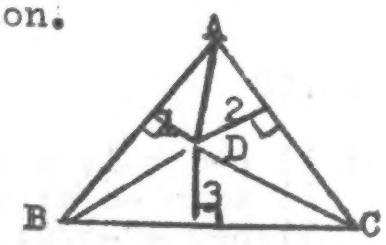
64.1.5.28

$$43/30 = 1 + 13/30 = 1 + 1/(30/13) = 1 + 1/(2 + 1/(13/4)) = 1 + 1/(2 + 1/(3 + 1/4))$$

64.2.6.13

There is a theorem (somewhere I am told) that gives the area in terms of the three given dimensions. You should be able to discover that theorem after this demonstration.

Let AC = s



ABC + BCD + CAD = ABC

$1/2(3s + 2s + 1s) = (s^2\sqrt{3})/4$

And $s = 4\sqrt{3}$. QED

64.3.5.59

$$-(-(-2)-(-2)^{-2})^{-(-2)} =$$

$$-(2 - 1/(-2)^2)^2 = -(2 - 1/4)^2$$

64.4.6.19

The number of selections of 6 points taken 3 at a time $(6!/(3!3!) = 20)$ minus the "triangles" AFC, ADB, BEC yields the answer 17. Which is the total number of named triangles (i.e., vertices given by a letter A,B,C,D,E or F); but I see quite a few unnamed triangles in my diagram. Betcha there were a lot of appeals on this one.

64.5.6.42

How quick is your re-action time? Using "standard" techniques, you should end up with $-3/8 < n < -3/2$, a lovely absurdity!

That's because the first step goes something like this: $(n-3)/3n < 3$ implies $n-3 < 9n$. But what if n represents negative values (which in this case is just what our author has shrewdly arranged)? Then there would be two possibilities: either $n-3 < 9n$ or $n-3 > 9n$. The latter case yields the upper bound of $b = -3/8$.

64.6.5.47

Divisibility by 11:

$(5+x+x) - (0+8) = 11q$

$2x-3 = 11q$

Therefore, $x = 7$.

And $(99)(513) = 50787$

64.7.6.16

My answer ain't 70 --- either(?). Mine is 28. Here's how I get it.

1) $6{s_1}^2 = (5/4)(6s_2)^2$

2) ${s_2}^3 = (1-x){s_1}^3$

From 1) ${s_1}^3 = (5/4)^{3/2}{s_2}^3 = (5\sqrt{5}{s_2}^3)/8$. Therefore, $1 = (1-x)(5\sqrt{5}/8)$. And $x = 1 - (8\sqrt{5})/25 = 1 - (32\sqrt{5})/100$

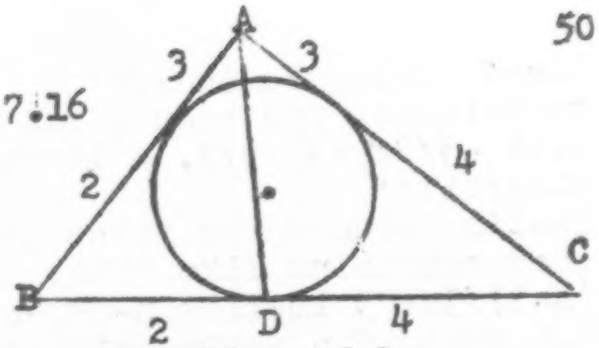
$x \approx 1 - .7168 = .28$

This time, if you don't like my approach, try a self-addressed stamped envelope to:

Mr. Gene Devereaux
3 Cardinal Drive
Woodstock, N.Y. 12498

He used to be Executive Secretary of the League, and is lots smarter than me. If there's a way --- an interpretation --- that yields that 70 answer, he'll know it.

64.8.7.16



Segments are obtained by fakeroo --- but the process can be algebraicized thusly:

$$\left.\begin{array}{l} x+y = 7 \\ x+z = 5 \\ y+z = 6 \end{array}\right\} \rightarrow \quad ; \left.\begin{array}{l} x+y = 7 \\ x-y = -1 \end{array}\right\} 2x = 6$$

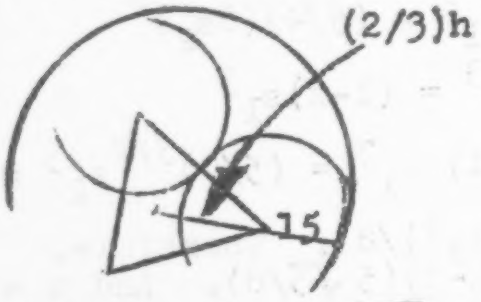
Law of Cosines: (twice)
$\cos(C) = (36+49-25)/(2(6)(7))$
And
$AD^2 = 16+49-2(4)(7)(5/7)$

64.9.5.59

$x^2 + |x| = 30$. Therefore,
1) $x^2 + x = 30$ for $x > 0$ and
$x = \{\cancel{-6}, 5\}$
2) $x^2 - x = 30$ for $x \angle 0$ and
$x = \{\cancel{6}, -5\}$

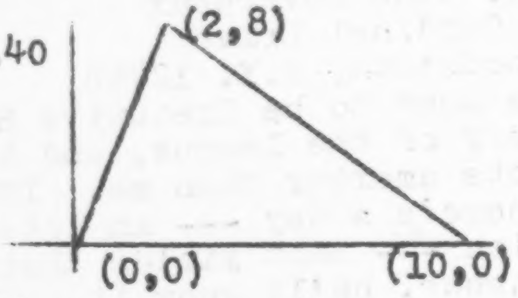
64.10.6.19

Equilateral triangle: 30-30-30



$h = (1/2)(30)\sqrt{3} = 15\sqrt{3}$
$(2/3)h = 10\sqrt{3}$
Radius of large circle = $15 + 10\sqrt{3}$. Find the area.

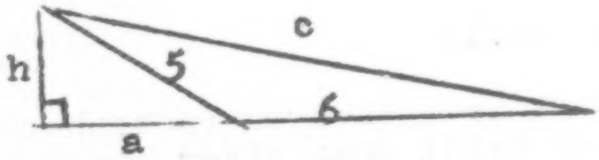
64.11.5.40



Criteria:
1) $y < 4x$, $0 < x \leq 2$
2) $y < -x + 10$, $2 < x < 10$

From 1), (1,1-3) : 3
(2,1-7) : 7
From 2), (3,1-6) : 6
(4,1-5) : 5 , etc
Therefore,
3+7+6+5+4+3+2+1

64.12.6.23



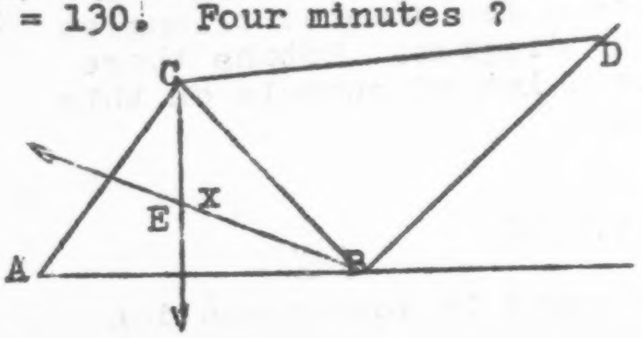
Area = $(1/2)(6)h = 9$; $h = 3$
Therefore $a = 4$ (3-4-5 right triangle)
Therefore $c^2 = 10^2 + 3^2$

64.13.5.08

Given: 1) $p-q = 1$; $p = q+1$
2) $pq = -6/m$; $m = -6/pq$
3) $p+q = -5/m$; $p+q = 5pq/6$
1) into 3) yields:
$2q + 1 = (5/6)(q^2 + q)$
Solve for q's, then for p's; then back into 2) yields the two values of m.

64.14.4.60 (Picture below)

1) 180 = angle A + angle B + angle C.
EXTERior angle C = angle A + angle B; Exter angle B = angle A + angle C.
(1/2)(EXT C + EXT B) = A + (1/2)(B + C).
Since in triangle BCD
(1/2)(EXT C + EXT B) + 50 = 180,
2) 130 = A + (1/2)(B+C)
Subtracting 2) from 1),
50 = (1/2)(B + C)
However, in triangle BCE,
(1/2)(B + C) + x = 180. Finally x = 130. Four minutes ?



64.15.6.31

First note that 9/10 and 7/9 are both less than 1. Let $(9/10)^{1/2} = x$ and $(7/9)^{1/4} = y$. Answer the question: which is greater, x^4 or y^4 ?
$x^4 = .81$; $y^4 = .77$
Therefore $x^4 > y^4$, and $x > y$.
Now,
Let $(4/3)^{1/2} = z$,
$(3/2)^{1/3} = w$.
$z^6 = 64/27 = 2+(10/27)$
$w^6 = 9/4 = 2+(1/4)$. Therefore $z^6 > w^6$ and $z > w$.
Therefore $y < x < w < z$.
You might notice that my answer is not the same one given in the answer key ---which I have found written twice in my archives. Either someone has been fostering "un gros canard" for several years --- or my reasoning above is all wet. It's your choice.

64.16.3.71

In the first quadrant, the abscissas are all positive; in the second quaerant, they're all negative. Due to symmetry, their sums are 0.

64.17.6.04

$\sqrt{x}(x^{3/2} + y^{3/2}) = 81$

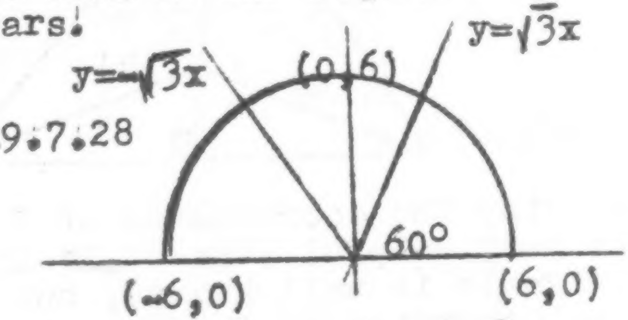
$\sqrt{y}(y^{3/2} + x^{3/2}) = 162$

$\sqrt{y}(81/\sqrt{x}) = 162$; $y = 4x$.
Go.

64.18.5.63

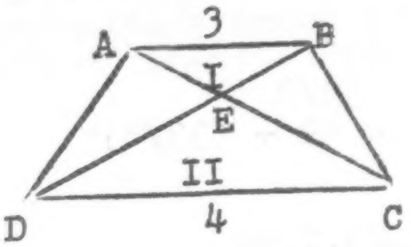
Algebra:
$(n+(n+1)+...+(n+(n-1)))/n = 31$
Numerator is the sum of an arithmetic sequence: therefore, $(n/2)(n+(n+(n-1))) = 31n$
Solving, $n = 21$.
Fakeroo Easy Way:
Middle term of the sequence is the average of the sequence; therefore, 31 is in the middle. Work to the left until the first term of the sequence appears.

64.19.7.28



Note: the graph of $y^2 = 3x^2$ consists of two straight lines. Since the slope is $\sqrt{3} = \tan(x)$, $x = 60°$. Therefore, the angle of the sector is 60°.
Area of Region = $(1/6)\pi(36)$.

64.20.6.20



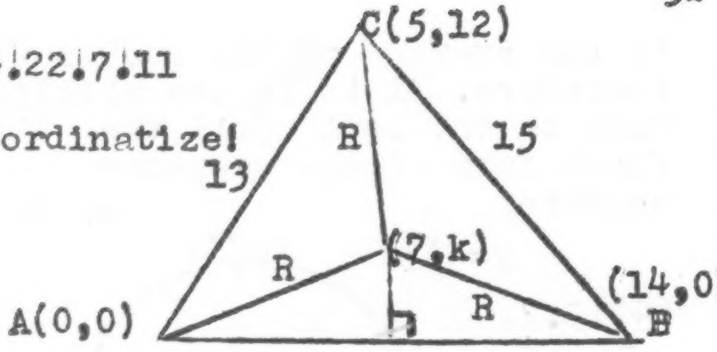
Since triangles AEB and CDE are similar, I/II = 9/16 = $3^2/4^2$. I + II = 50.
Therefore, I = 18, II = 32.
And $18 = (1/2)h_1(3)$
$32 = (1/2)h_2(4)$
But h_1+h_2 = altitude of the trapezoid. QED

64.21.6.34

$n = 8m+5$; $n = 13q+7$; Therefore $13q-8m+2 = 0$, or $13q + 2 = 8m$
Modulo 13: $8m = 2$;
$m = 1/4 = -12/4 = -3 = 10$
And, $m = 13k+10$
Therefore, $q = 8k+6$ (by substitution). For $k = 0$, n is a multiple of 5. For $k = 1$, $m = 23$, $q = 14$, $n = 189$.
OR, as I'm certain most Mathletes did, start listing the possible values in two separate columns. Be observant.

64.22.7.11

Coordinatize!



Knowing the coordinates of C is very helpful (the 13-14-15 triangle is well known), but can otherwise be found most readily using the distance formula twice; i.e.,
assume C(x,y). Then
1) $x^2 + y^2 = 169$
2) $(x-14)^2 + y^2 = 225$
Subtracting 1) from 2) immediately yields $28x = 140$.
k is found similarly; i.e.,
$(7-0)^2 + (k-0)^2 = (7-5)^2 + (k-12)^2$
$k = 33/8$.
$R^2 = 49 + (33/8)^2$

64.23.7.04

Well, I got 18 of 'em anyway!

$7(10a+b) = 100c + 10d + e$
$70a + 7b = 100c + 10d + e$
$a + b = c + d + e$
Subtracting,
$69a + 6b = 99c + 9d$
$23a + 2b = 33c + 3d = 3(11c+d)$.
This last statement indicates that 23a + 2b must have a factor of 3 as a necessary (but not sufficient) condition that a+b have the same sum as c+d+e.

I made up the following sieve, eliminating those numbers which did not satisfy the given conditions (an interesting rule appears after awhile).
Necessary Condition: 23a + 2b be divisible by 3.
Let a = 1; then b = 2,5 or 8 Reject $1\underline{2}$
Let a = 2; then b = 1,4 or 7 Reject all three.
Let a = 3; then b = 0,3,6 or 9
Let a = 4; b = 2,5,8 Reject $3\underline{2}$
Let a = 5; b = 1,4,7. Reject all three.
Let a=6; b = 0,3,6,9
a = 7; b = 2,5,8 Reject $7\underline{2}$
a = 8; b = 1,4,7 Reject all three
a = 9; b = 0,3,6,9

Want to know the two missing numbers? Want to know how to do this problem in 7 minutes? Correctly? Try sending a self-addressed stamped envelope to:

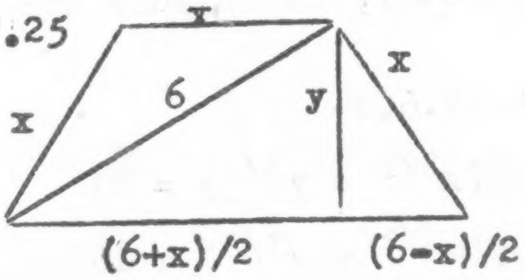
Mr. Harry Sitomer
36 Amherst Court
Huntington, N.Y.

He's the person who made up all these problems; and if he can't do it, well ......

64.24.5.A

Sum of two perfect squares will be zero only if each is zero.
Therefore both $y-x-2$ and $y - x^2$ must be zero.
Solving for y in each case and equating the results yields the quadratic $x^2 - x - 2 = 0$.
First solve for x and then find the corresponding y's.

64.25.7.25



$y^2 = x^2 - ((6-x)/2)^2$ and
$y^2 = 36 - ((6+x)/2)^2$
Eliminate the y's and solve for x. No sweat.

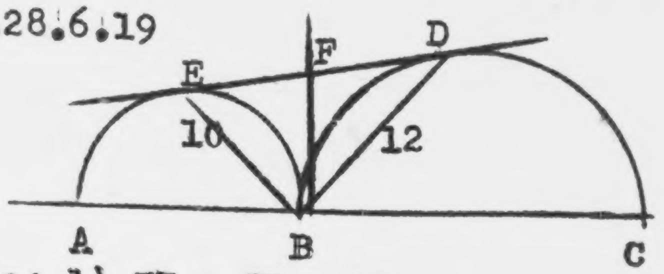
64.26.4.76

Length of tunnel: x
Distance travelled during alloted time: x+100
Therefore, $(40)(1760)(1/2) = x + 100$. Solve.

64.27.6.46

c = 2 is given. −b/2a = 1
Therefore, 2a = −b.
9a + 3b + 2 = 0
9a + 3(−2a) + 2 = 0. Solve.

64.28.6.19

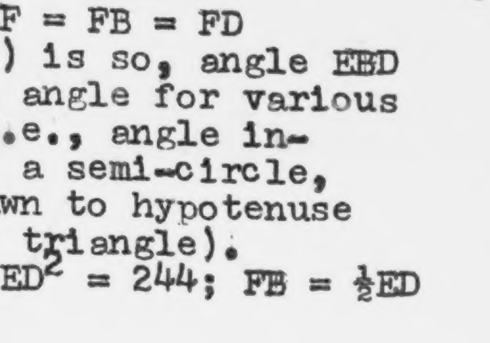


Note: 1) EF = FB = FD
2) Since 1) is so, angle EBD is a right angle for various reasons (i.e., angle inscribed in a semi-circle, median drawn to hypotenuse of a right triangle).
Therefore $ED^2 = 244$; FB = ½ED

64.29.5.36

Given: p/q = m/n
p+q = −2/3; m+n = −1; mn=1 ;
pq = a/9.
If p/q = m/n, then (p+q)/q = (m+n)/n; So (−2/3)/q = −1/n .
q = (2/3)n = 2/(3m)
Since p/q = m/n, pq = mq^2/n.
Finally,
pq = a/9 = (m/n)(2n/3)(2/(3m))
Solve for a. Five minutes ??

64.30.7.17



Triangles EGA and BGF are similar. Ratio of altitudes is 2/3. Therefore altitude onto AE of triangle AGE is 12.
Areas: AGE = 60; ABF = 225
HEF = 75; Rectangle HFBA = 450
Therefore, FGE = 90.

1964 - 1965

65.1.5.39

I will give you my route: but no reasons. They're all yours.

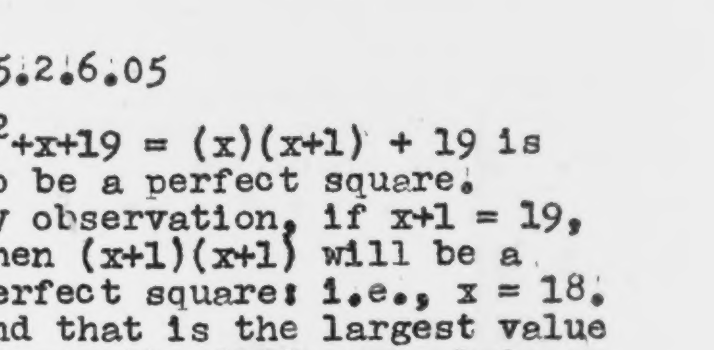
$$\begin{array}{r} abc \\ \underline{d7} \\ 3ef4 \\ \underline{5ghj} \\ klmn4 \end{array}$$

1) c = 2 (Hurray!)
2) a = 4 or 5
3) a definitely = to 5, d = 9
4) 0 b 6
5) b = 6
6) Go!

65.2.6.05

$x^2+x+19 = (x)(x+1) + 19$ is to be a perfect square. By observation, if x+1 = 19, then (x+1)(x+1) will be a perfect square: i.e., x = 18. And that is the largest value of x: the difference between any two squares gets much greater than 19 + the previous number to be squared. That is, 400 − 361 = 39 > 19+19 . Therefore, merely search out all the squares below 18^2 = 324.
289 − 19 = 270 ≠ product of two consecutive integers.
49 − 19 = 30 = (6)(5)
25 − 19 = 6 = (3)(2)
Therefore, 2,5,18 .

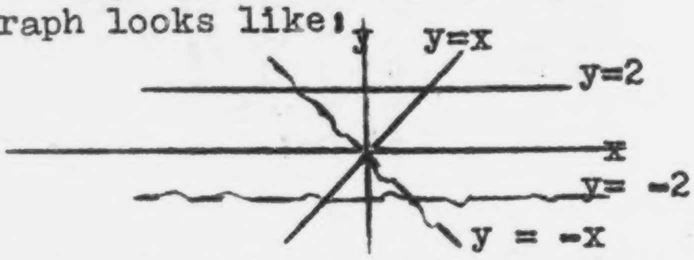
65.3.7.01



Coordinates of B are (5,12) (Remember?). Coordinates of D are (14 − ((14−5)/2),12/2) Or D(19/2,6). Since triangles AEF and AGD are similar, AEF/AGD = AE^2/AG^2 = 25/(361/4) . And triangle AEF = (100/361)·(1/2)(19/2)(6) = 150/19

65.4.5.19

It's a look at! First term: $((x-y)(y-2))^2$ must be zero! Therefore either $y = x$ or $y = 2$. Two lines of values there!
Second term: $((x+y)(y+2))^2$ must also be zero!
Therefore either $y = -x$ or $y = -2$. Two more lines here.
Graph looks like:



There are 5 intersections, but only "squigley on smooth" count. Think about it.

65.5.4.37

P1P2, P1P3, P1P4, P1P5.
That is, by fixing one point P1, the other 4 vertices can be situated at the second point.

65.6.5.74

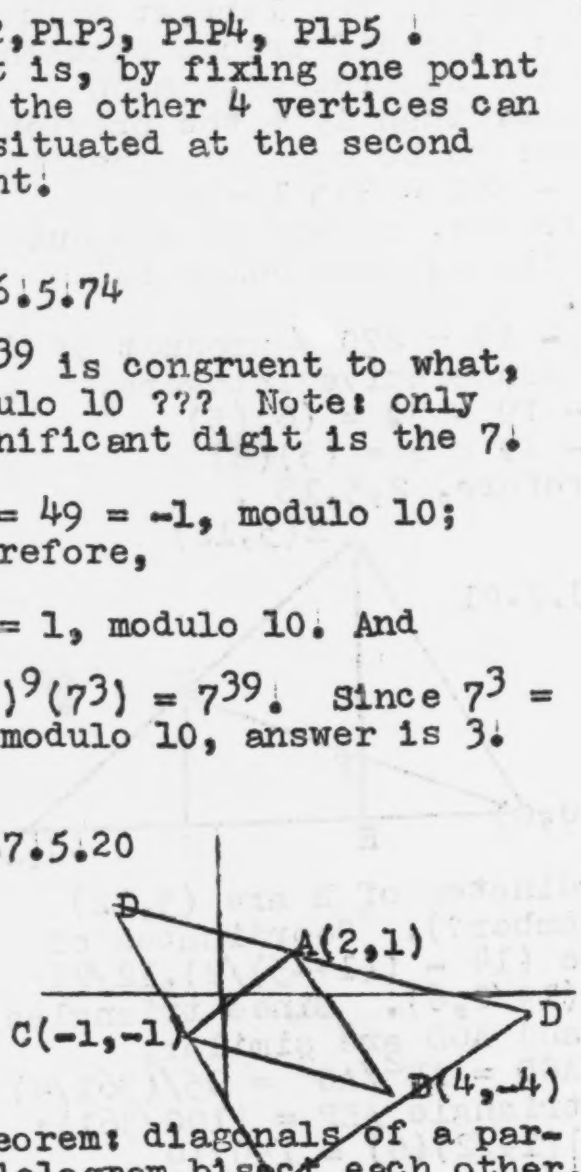
127^{39} is congruent to what, modulo 10 ??? Note: only significant digit is the 7.

$7^2 = 49 = -1$, modulo 10;
Therefore,

$7^4 = 1$, modulo 10. And

$(7^4)^9(7^3) = 7^{39}$. Since $7^3 = 3$, modulo 10, answer is 3.

65.7.5.20



Theorem: diagonals of a parallelogram bisect each other.

If M stands for midpoint, then $M(AB) = (3,-3/2)$; which then sez that $D(7,-2)$.
$M(BC) = (3/2,-5/2)$ sez $D(1,-6)$.
$M(AC) = (1/2,0)$ sez $D(-3,4)$.

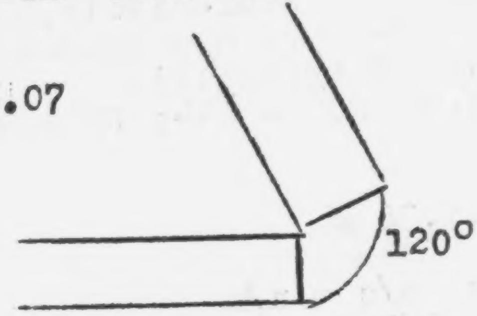
65.8.5.58

$\log_{x^2}(2) + \log_4(x) = 1$

$(\log_2 2)/(2\log_2 x) + (\log_2 x)/\log_2 4$
$= 1/(2\log_2 x) + (\log_2 x)/2 = 1$

$(\log_2 x)^2 - 2\log_2 x + 1 = 0$
$\log_2 x - 1 = 0$
Or:
By fakeroo --- but watch out for second answers.

65.9.5.07



Area of triangle = $36\sqrt{3}/4$
Three rectangles = 36
3 sectors = $3(1/3)\pi(4)$.
Sum up and compute.

65.10.4.24

$x+1 = 1/((1+x+1)/(1+x)) =$
$(1+x)/(x+2)$
If $x \neq -1$, $x+2 = 1$, $x = -1$.
WHA ???

Therefore, none!

65.11.6.61

Simple way: Compute and divide by 26
Clever way: Factor, obtaining

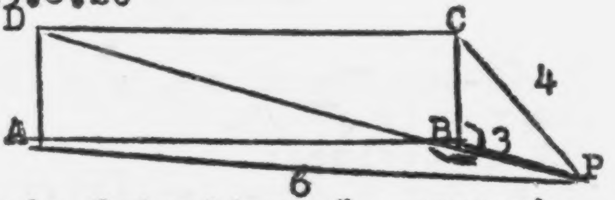
$(3^4 + 2^4)^2 = (81 + 16)^2 =$

$97^2 = (-7)(-7)$, modulo 26;
And $49 = 23$, modulo 26.

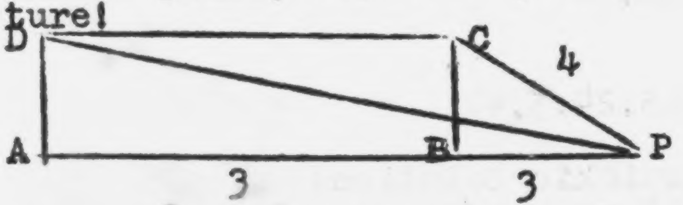
65.12.6.19

B's race: $(r1)(t1) = 5250$;
$r1 = 5250/t1$
C's race: $(r2)(t1) = 5190$;
$r2 = 5190/t1$
Let t2 be the time for B to finish the race. Then
$t2 = 30/(5250/t1) = (30t1)/5250$
In time t2, C goes x; and
$t2 = (xt1)/5190$.
Therefore, $30/5250 = x/5190$
Final answer: $90 - x$.

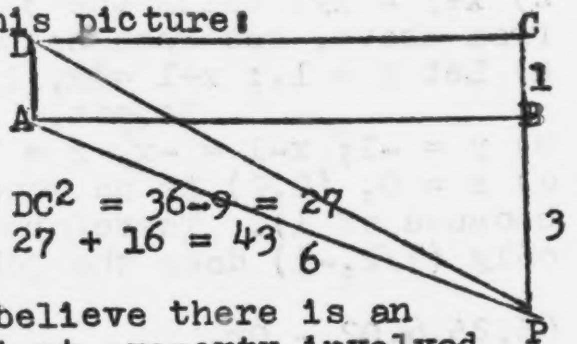
65.13.6.20



"Cosmic Coincidence" approach:
Assume P is a point directly over B in 3-space.
Then $BC^2 = 16-9 = 7$
$AB^2 = 36-9 = 27$ and
$DB^2 = 27+7 = 34$
Finally, $DP^2 = 34+9 = 43$.
How's that grab ya ?
OR
In the plane, try this picture!



Then $CB^2 = 16-9 = 7$
$DP^2 = 36 + 7 = 43$
Or
Try this picture:



$AB^2 = DC^2 = 36-9 = 27$
$DP^2 = 27 + 16 = 43$

I do believe there is an invariant property involved here.

65.14.5.38

$(3x+x)x + x = 39$. Go to it.

65.15.7.07

I have no idea how to attack this one! Oh, I coordinatized a right triangle with legs of length 10 and 12 and found (after much diligent labor) the given answer.

But that leaves me cold. It is fairly obvious from the statement of the problem that there is an invariant relationship involved --- so help yourself; right triangle, isosceles, 30-60-90, equilateral --- whichever is your choice, whatever will do the the job for you.
But to show that for a triangle of given area (i.e., whose base and altitude are inversely proportional) that the ratio of the segment of the median to a side not a base cut off by the lines trisecting the base drawn from the opposite vertex to the median in question is an invariant --- now, that's got me hung up!

65.16.5.17

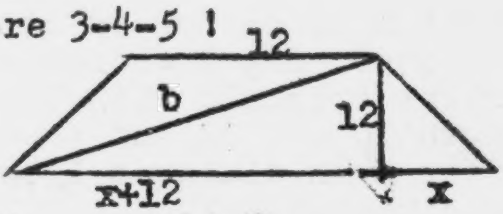
$1/7 = .\overline{142857}$; $6/7 = .\overline{857142}$
$2/7 = .\overline{285714}$; QED
(I understand this can be done algebraically also.)

65.17.6.49

Man's rate: x mph. Time is the same.
Man: $(x)(t1) = 2d/5$
Train: $60t1 = k$
Therefore, $(2d/5)/x = k/60$
$24d = kx$
Second choice:
Man: $(x)(t2) = 3d/5$
Train: $60(t2) = k+d$
Therefore, $(3d/5)/x = (k+d)/60$
$36d = kx + dx$
$36d = (24d) + dx$. Solve for x.

65.18.6.29

One More 3-4-5 !

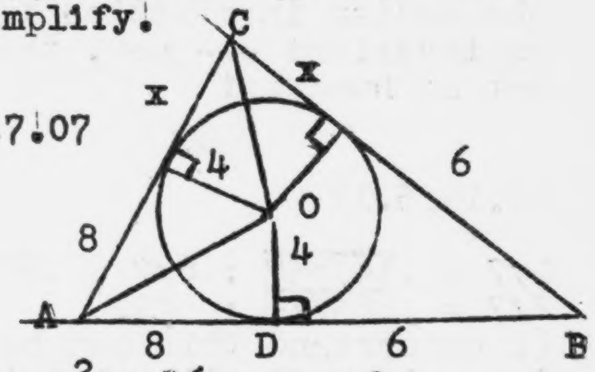


$(12+x)^2 + 12^2 = b^2$ and
$b = 12 + 2x$; $b^2 = (12+2x)^2$
Therefore, $(12+x)^2+144 = (12+2x)^2$
Solve for x. Go.

65.19.6.07

$x^{10}+1 = x^5z$; $z = x^5 + 1/x^5 =$
$(x+1/x)(x^4 + 1/x^4 -(x^2 + 1/x^2) + 1)$
But $x^2+1 = xy$ means $y = x + 1/x$
And $y^2 = x^2+2+(1/x^2)$;
$y^2 - 2 = x^2 + (1/x^2)$
$(y^2 - 2)^2 = x^4 + 2 + (1/x^4)$
Finally,
$y^4-4y^2+2 = x^4 + 1/x^4$
Now substitute:
$z = (y)(y^4-4y^2+2 -(y^2-2) + 1)$
And simplify.

65.20.7.07

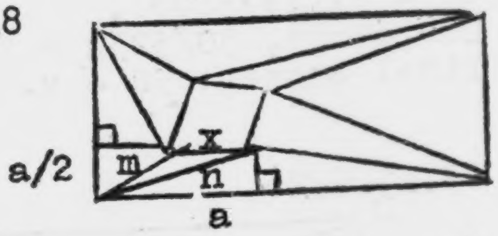


$(CO)^2 = x^2 + \underline{16 \text{ ; } x} = ?$
$\cos(y) = 8/(\sqrt{64+16}) = (2\sqrt{5}/5)$
$\cos(2y) = 2 \cos^2(y) - 1$
$= 8/5 - 1 = 3/5$
(This should give you the hint that it's the good ole 13-14-15 triangle again; i.e., a 9-12-15 back to back with a 5-12-13).
Law of Cosines:
$3/5 = (196+(8+x)^2-(6+x)^2)/(2)\cdot(14)(8+x)$. Solving for x is no major problem. Ho-Ho-Ho. (x = 7, just as I intimated above.)

65.21.5.30

Rate A = 1/4 candle/hour
Rate B = 1/5 candle/hour
(1/4)t = k1, gone. 1-k1 remains
(1/5)t = k2, gone. 1-k2 remains.
When will
1-k2 = 4(1-k1) ????
When k2 = 4k1 - 3. And since
k1/k2 = 5/4, k1 = 15/16 .
t = 4(15/16) = 15/4 hours.

65.22.7.18



30-60-90: Therefore, $m = (2a\sqrt{3})/3$ and $n = (a\sqrt{3})/3$.
Law of Cosines:
$x^2 = (4a^2 + a^2 - 2a^2\sqrt{3})/3$
QED

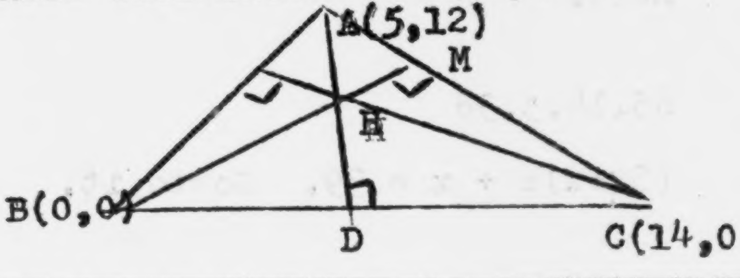
65.23.5.29

$(x^2+cx+d)(x^2+cx+d) =$
$x^4 + (2c)x^3 + (2d+c^2)x^2 + (2dc)x + d^2$
First, c = 4; then d = -4.
Go.

65.24.5.42

Quickie Solution:
1) $xy = x/y$; $y = 1$, $y = -1$, $x = 0$ three lines
2) $x+y = xy$; using the 3 lines from above, see what happens!
a) Let y = 1.; x+1 = x, 1 = 0 Reject.
b) y = -1; x-1 = -x, x = 1/2.
c) x = 0, (0,0) is no good because of 1). Therefore only (1/2,-1) does the job.

65.25.6.02 - Or,
More Coordinate Geometry:

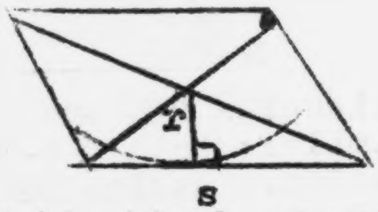


slope AC = -12/9 = -4/3
Line BM: y = (3/4)x
Line AD: x = 5
Therefore, H(5,15/4)
AH/HD = (33/4)/(15/4) = k/5
Compute.

65.26.5.01

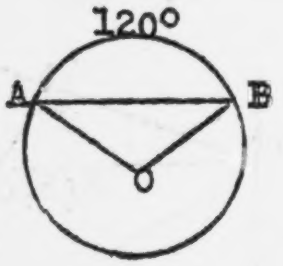
Assume someone desires to purchase 40 yards of cloth at a dollar a yard. He will pay for 40 yards, but receive only 35 yards. The dishonest profit is therefore $5, or 1/7 of cost. The problem states that this dishonest profit ((1/7)c) plus the honest profit (p) is equal to 1/4 of sales, where sales are 8/7 of costs.
Therefore, (c/7 + p) = (1/4)·(8c/7)
And p = c/7 (Splits his profit evenly - half honest, half dishonest.)
Likewise,
(c/9 + p) = (x/7)(10c/9), where it is assumed p still to be taken as c/7. (Keeps his honest profit the same even though his conscience is starting to bother him.)
Therefore, (1/9)+(1/7) = (x/7)(10/9). x = 1.6 .

65.27.5.78



1) Area = (1/2)(d1)(d2) = 100
2) Area = (4)(1/2)(sr) = 100
3) $s^2 = 5^2 + 10^2 = 125$
From 2) r = 50/s. Go.

65.28.5.05



Area of sector AOB = $\pi r^2/3$
Area of triangle AOB = $\dfrac{r^2\sin(120)}{2}$
Area of lune AB = $r^2(\pi/3 - \sqrt{3}/4)$
Area of Circle - Lune =
$r^2(\pi - (\pi/3) + (\sqrt{3}/4))$.
Ratio should become:
$(8\pi + 3\sqrt{3})/(4\pi - 3\sqrt{3})$.
Go.

65.29.6.29

Expand and simplify. Same as problem 62.24, only simpler.

65.30.6.08

$x^2 - (c-3)x - 3c$
(x-c)(x+3)
And
$x^2 + 2cx + c^2-1$
(x+c+1)(x+(c-1))

Therefore:
c+1 = -c ; c = -1/2
Or
c+1 = 3 ; c = 2
Or
c-1 = -c ; c = 1/2
Or
c-1 = 3 ; c = 4

1965 - 1966

66.1.6.07

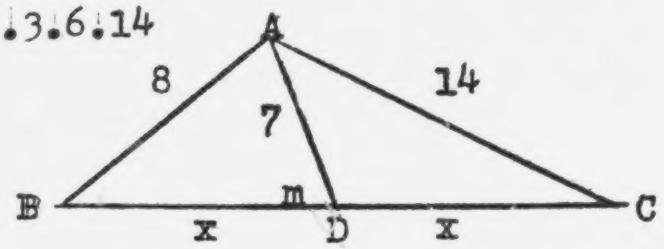
Since "1" was dictated for the number to occupy the hundreds position, it was not eligible for the other two places. Neither were any even digits nor the digit 5, since one of the pair would not be prime. This left then only 3,7,9 to fill the positions and the pairs 137,173; 139,193; 179,197; were eligible. Test away.

66.2.6.12

$(1/r^3)+(1/s^3) = (r^3+s^3)/(rs)^3 =$
$((r+s)(r^2+2rs+s^2 - 3rs))/(rs)^3 =$
$((r+s)((r+s)^2 - 3rs))/(rs)^3$.

Since r+s = 5 and rs = 3, the answer follows directly.

66.3.6.14

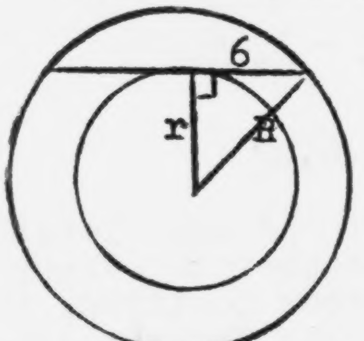


Law of Cosines:
$64 = 49 + x^2 - (2)(7x)(\cos(m))$
And
$196 = 49 + x^2 -(2)(7x)\cos(180-m)$
$= 49 + x^2 +(2)(7x)\cos(m)$
Take the two equations together and solve.

66.4.5.27

Again, average Rate ≠ average of rates.
Average Rate = (Total distance) divided by (Total time).
$wt_1 = 1$; $rt_2 = 1$; average rate $=(1/3)r$
Therefore, $r/3 = (1+1)/(t_1+t_2)$.
Substitute in for t_1 and t_2 and find the required ratio.

66.5.5.34



Area of ring = $\pi(R^2 - r^2)$;
But $R^2 - r^2 = 36$. QED

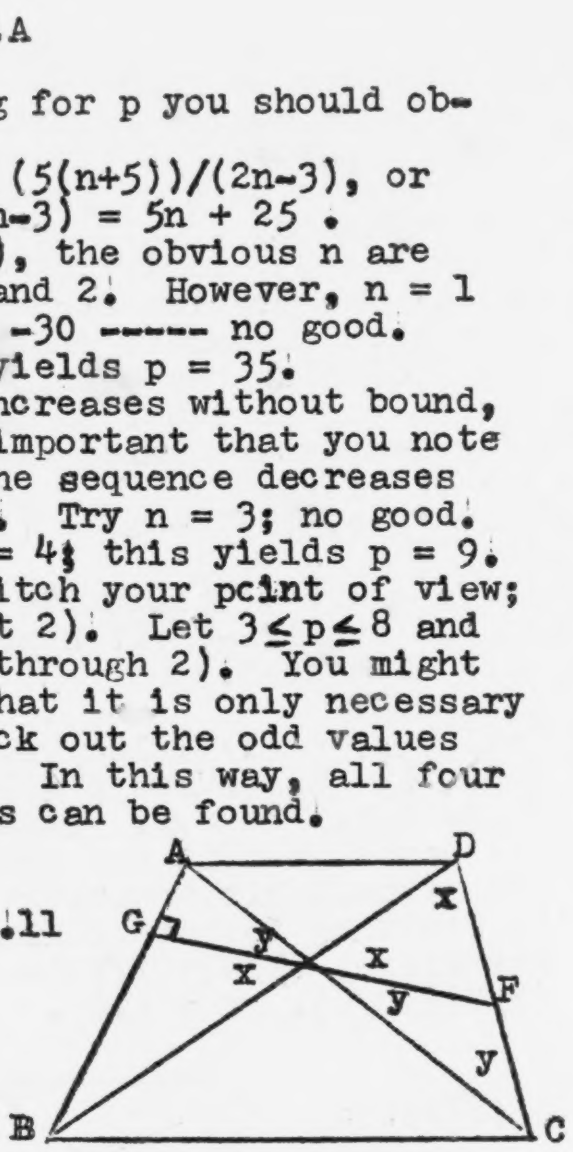
66.6.6.28

There are found to be 68 available horizontal spaces and 55 vertical spaces upon totaling each direction and deducting from 225. Using x for 11-letter words, y for 10=letter words and b for blanks, the Diophantine system yields $11x + b = 68$, $10y + b = 55$ and finally $11x - 10y = 13$. Since x,y are integers such that $0<x\leq 6$, $0<y\leq 5$, the only suitable values of x and y are 3 and 2 respectively. Solve for b.

66.7.6.A

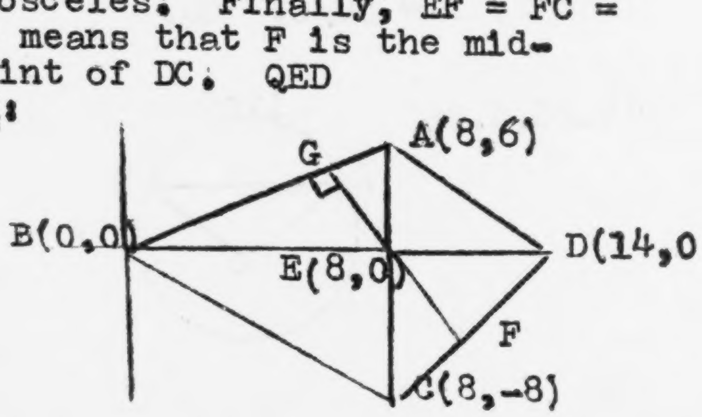
Solving for p you should obtain
1) $p = (5(n+5))/(2n-3)$, or
2) $p(2n-3) = 5n + 25$.
From 1), the obvious n are n = 1 and 2. However, n = 1 yields -30 ----- no good. n = 2 yields p = 35.
As n increases without bound, it is important that you note that the sequence decreases to 5/2. Try n = 3; no good. Try n = 4; this yields p = 9. Now switch your point of view; look at 2). Let $3\leq p\leq 8$ and check through 2). You might note that it is only necessary to check out the odd values for p. In this way, all four answers can be found.

66.8.6.11



Triangles BEG and CDE are similar: both 3-4-5 right triangles. Angles BEG, CDE, and DEF are congruent. Therefore triangle DEF is isosceles. Similarly, triangle CEF is isosceles. Finally, EF = FC = DF means that F is the midpoint of DC. QED

OR:



Slope AB: 3/4; slope EG: −4/3
Line EG: $4x+3y = 32$
slope DC: 4/3;
Line DC: $4x - 3y = 56$
Therefore, $8x = 88$; and
F(11,−4) Note that (11,−4)
is the midpoint of CD. QED.

66.9.6.30

$$\begin{array}{rrrrr}
1 & -6 & 7 & -5 & 2 \\
 & 2 & -8 & -2 & \\
\hline
1 & -4 & -1 & -7 & \\
 & 2 & -4 & & \\
\hline
1 & -2 & -5 & & \\
 & 2 & & & \\
\hline
1 & 0 & & &
\end{array}$$

Therefore, $x^3 - 6x^2 + 7x - 5 = (x-2)^3 - 5(x-2) - 7$.

66.10.6.28

More Way Out !
Given sequence:
462, 562, 670, 780, 894, 1012
First difference sequence:
100, 108, 110, 114, 118
Second difference sequence:
8 , 2 , 4 , 4
Therefore 108 should be 106 .
Working backwards, the correct number should be 564.

66.11.5.13

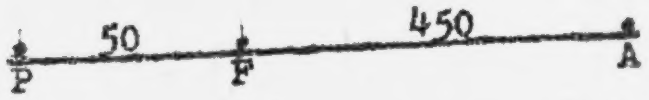
There will be 24 such four-digit numbers each having the same distinct integers in some order. Each digit (a,b,c,d) will appear in each decimal position 6 times. The sum can therefore be expressed as 6666(a+b+c+d). This yields (a+b+c+d) equal to 29 and a quick trial will do the trick.

66.12.5.06

Follow the bouncing ball:

1) $(ax)^{\log a} = (bx)^{\log b}$
2) $\log a(\log ax) = \log b(\log bx)$
3) $\log^2 a + (\log a)(\log x) = \log^2 b + (\log b)(\log x)$
4) $\log x(\log a - \log b) = \log^2 b - \log^2 a$
5) $\log x = -\log a - \log b = -\log ab = \log(ab)^{-1}$
6) Thus $x = 1/ab$

66.13.6.03



In "ellipse talk", $2a = 500$, $a = 250$, $c = 250 - 50 = 200$. Since $b^2 = k^2 = a^2 - c^2$, $k^2 = (50)^2(5^2 - 4^2)$.

66.14.6.16

Rationalize $6/(\sqrt{5} + (\sqrt{3} + \sqrt{2}))$ by $(\sqrt{5} - (\sqrt{3} + \sqrt{2}))/(\text{itself})$, getting $(6(\sqrt{5} - \sqrt{3} - \sqrt{2}))/(5 - (\sqrt{3} + \sqrt{2})^2)$, which reduces to $(6(\sqrt{5} - \sqrt{3} - \sqrt{2})/(-2\sqrt{6})$. Keep going.

66.15.6.40

Let the sides of the smaller triangle be 8,x,y and the larger x,y,27. The proportions $8/x = x/y = y/27$ lead to $x^2 = 8y$, $xy = (8)(27)$. Therefore $(x)(x^2/8) = (8)(27)$. Solve.

66.16.5.36

If d represents the length of one rail and x the speed of the train in miles per hour then an equality in feet per second can be established such as $(dx)/15 = (5280\ x)/3600$. Solve for d.

66.17.6.00

If $(a+b-c)/c = (b+c-a)/a$, then $a^2+ab-ac = bc+c^2-ac$, or, if you don't immediately simplify,

$(a^2-ac) + (ab-bc) + (ac-c^2) = 0$

And $(a-c)(a+b+c) = 0$.

Therefore, either $a = c$ or $a+c = -b$.

But $a=c$ implies $b=a=c$.

Therefore the given expression reduces to 8.

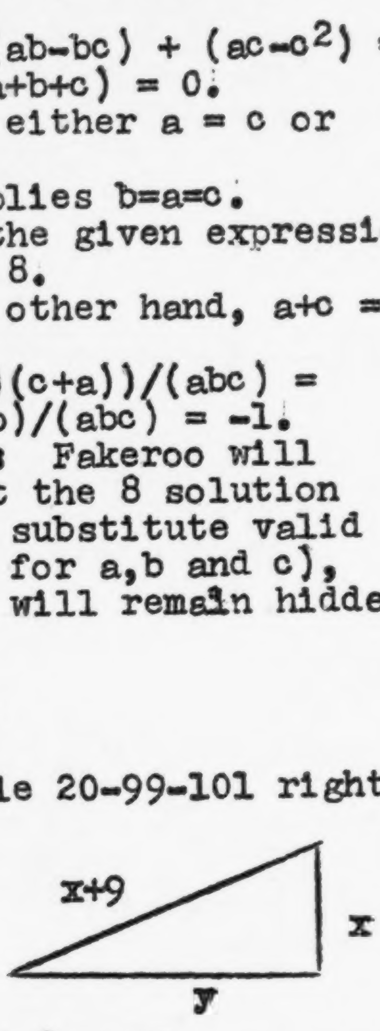
If, on the other hand, $a+c = -b$, then

$((a+b)(b+c)(c+a))/(abc) = (-c)(-a)(-b)/(abc) = -1$.

By the way: Fakeroo will readily get the 8 solution here (just substitute valid numbers in for a,b and c), but the -1 will remain hidden.

66.18.7.08

The good ole 20-99-101 right triangle!



1) $2x + y + 9 = xy$

2) $y^2 = 18x + 81$

From 1) and 2) comes 3), namely,

3) $18x^3 + 41x^2 - 180x = 0$

or $(9x-20)(2x+9) = 0$

Note: To factor 3) is nigh impossible; however, if you use $b^2 - 4ac = 41^2 + 12,960 = 14,641$, you should immediately recognize the latter number as 11^4.

66.19.6.A

Since $b^2-4ac = 0$ must be so,

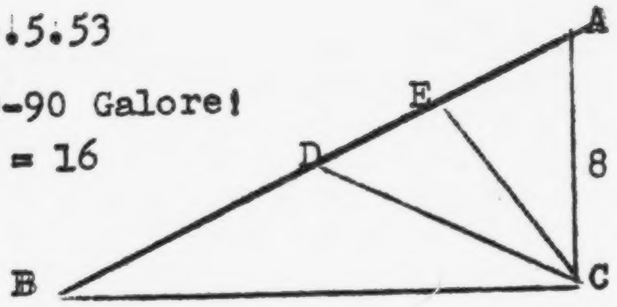
$4 - 4(c+d\sqrt{3})(7+4\sqrt{3}) = 0$.

Therefore, $4 = 28c + 48d$ and $7d + 4c = 0$. Solve.

66.20.5.53

30-60-90 Galore!

$AB = 16$



By observation, angle $B = 30^\circ$

Therefore, $BD = CD$.

Also, CE is perpendicular to AB; therefore, $EA = 4$ and likewise, $DE = (1/2)CD = (1/2)BD$.

So: $16 = BD + (1/2)BD + 4$.

66.21.6.02

A sample solution on this was written up in 1966 --- which has perplexed me greatly. Primarily because it was all wet. I've also lost all the answers for the 1965 - 1966 year, and therefore my candidate of 119 was (intuitively) arrived at as follows:

When testing the number 95, it is necessary to cast out the odd multiples of 15 less than 95 (i.e., 15,45,75) in your search for a multiple of 8. For instance, $95 - 15 = (8)(10)$ does the job. Likewise, when testing 94, it is necessary to cast out the even multiples of 15 less than 94 (i.e., 30, 60, 90) in your search for a multiple of 8. For instance $94 - 30 = 64$. Now, 96 balls cannot be accommodated in the two boxes because when casting out the even multiples of 15, there just isn't enough lee-way. For instance, if you could cast out 120, obtaining -24, you'd be all set.

At any rate, the crux of my argument is that for any number of balls, after 120, there are a sufficient number of either odd multiples of 15 (4 of 'em --- 15,45, 75, 105) or even multiples of 15 (4 of 'em;

30, 60, 90, 120) to be cast out leaving a multiple of 8. However, before 120, no such luck (?). Therefore, 119 is my answer.

66.22.6.20

Rationalize the denominator, obtaining:
$((x+1)+(x-1)+2\sqrt{x^2-1})/(x+1-(x-1))$,
or $x+\sqrt{x^2-1} = a/b$.
Therefore, $\sqrt{x^2-1} = (a-bx)/b$.
Squareing and simplifying should lead to the expression $x = (a^2-b^2)/2ab$.

66.23.5.34

$x^2-8x+16+y^2-4y+4+z^2-6z+9 = 0$

$(x-4)^2 + (y-2)^2 + (z-3)^2 = 0$
Sum of three squares is zero only if each term is zero. $(x,y,z) = (4,2,3)$. Compute.

66.24.6.36

$\sum_{i=1}^{n} i^2 = (k)(\sum_{i=1}^{n} i)$ is given;

$(n(n+1)(2n+1))/6 = k(n(n+1))/2$
Therefore, $2n+1 = 3k$.

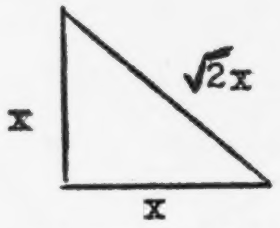
66.25.8.19

The sum of two sides of a triangle must be greater than the third side. Therefore:
1) $r^2 + r > 1$
2) $r^2 + 1 > r$
3) $r + 1 > r^2$
Since r must be positive,
1) yields $r > (-1+\sqrt{5})/2$
2) yields all values of r
3) yields $r < (1+\sqrt{5})/2$.
QED

66.26.6.12

List the known decimal primes - start with 11, end with 73, omit those ending with 9; translate the balance into other bases and eliminate.

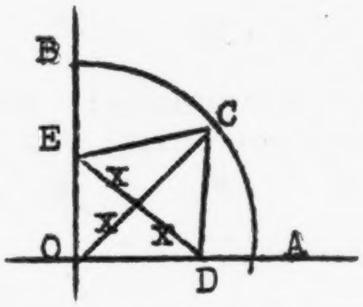
66.27.6.01



1) $x^2/2 = (3-2\sqrt{2})a^2$; $x^2 = (6-4\sqrt{2})a^2$
2) $x(2+\sqrt{2}) = ka$; $x^2 = (ka)^2/(2+\sqrt{2})^2$.
Therefore, $k^2 = (6+4\sqrt{2})(6-4\sqrt{2})$
$\sqrt{k^2} = \sqrt{4}$; $|k| = 2$; $k = \{\pm 2\}$
Note: Both k and a can be negative. Check it out.

66.28.6.01

h of ECD = $\sqrt{3}$ x



Side of triangle CDE = 2x.
Then, $x + x\sqrt{3} = r$; $x = -r(1-\sqrt{3})$ divided by 2. $2x = (\sqrt{3}-1)r$.
Area of quadrant $= \pi r^2/4$.
Area of CDE $=((4-2\sqrt{3})r^2\sqrt{3})/4$.
QED.

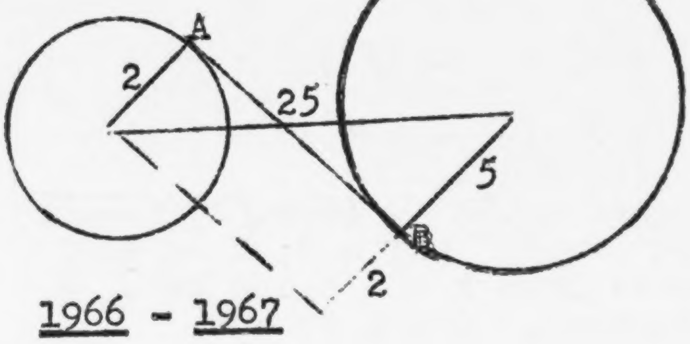
66.29.5.51

Using a "well-known" ratio theorem, it is evident that since $(a+b)/(b+c) = (c+d)/(d+e) = (e+f)/(a+f)$, then $(a+b+c+d+e+f)/(b+c+d+e+f+a) = (a+b)/(b+c) = 1$.
Thus it can be deduced that $a=c=e$ and $a/c + c/e = 2$.
Now you know and I know that 51% of our Mathletes didn't figure that out in 5 minutes, but what they did do was

probably something like this: Let $a = 1$, $b = 1$, $c = 1$ and $d = 1$; therefore $e = 1$. Substituting into $(ae+c^2)/ce$ gives 2.
Note that this "technique" does not always "work"; i.e., see 66.17 .

66.30.7.21

Regardez!



1966 - 1967

67.1.5.35

The concensus: start counting --- being very careful.

67.2.5.02

Set $a-x = c$; therefore $x=a-c$.
Set $b-x = c$; therefore $x=b-c$.
First substitute $x = a-c$ into the original; this yields $a = b+c-1$. Then, substitute $x = b-c$ into the original; this yields $a = b-c+1$.

67.3.6.29

Again, the Lamb approach:

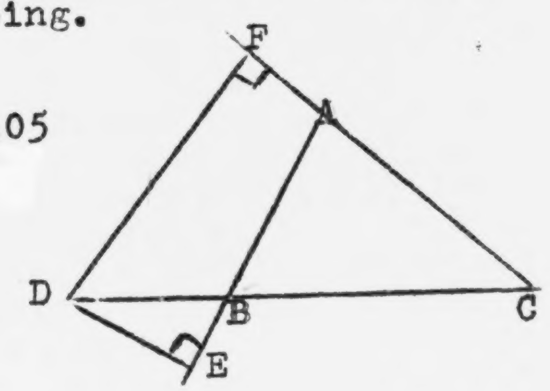
$$-1/2 \begin{vmatrix} 1 & 3 \\ 3 & 8 \\ 7 & 2 \\ 4 & -4 \\ 1 & 3 \end{vmatrix} = -1/2((1\cdot 8 + 3\cdot 2 + (7)(-4) + (4)(3)) - ((3)(3) + (8)(7) + (2)(4) + (-4)(1))) = -1/2(-2 + (-69)) = 71/2 .$$

67.4.6.39

$$x^4-9x^2+12x-4 = x^4 - (9x^2-12x+4) = x^4 - (3x-2)^2 = (x^2-(3x-2))(x^2+(3x-2)) = (x^2-3x+2)(x^2+3x-2);$$

keep going.

67.5.6.05



$DE = 4$; $DF = 7$; FAC is the case, sinve others lead to contradictions.
1) Triangle DEB is 30-60-90; $DB = 8\sqrt{3}/3$
2) Triangle DCF is 30-60-90; $DC = 14\sqrt{3}/3$
3) $BC = 2\sqrt{3}$. QED.

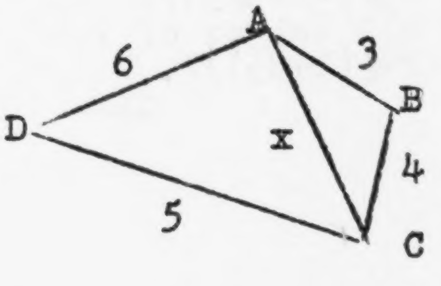
67.6.6.07

Strictly Fakeroo! The more you do the better you get --- is what I'm told.

67.7.6.06

Simplicity itself: Let $x = 1/(3+1/(3+1/(3+\ldots = 1/(3 + (x))$. That is, since the continued fraction is "infinite", replace it with itself after one term; solve the subsequent quadratic, rejecting the negative root.

67.8.7.07



Law of Cosines:

$\cos(B) = (9+16-x^2)/((2)(3)(4))$

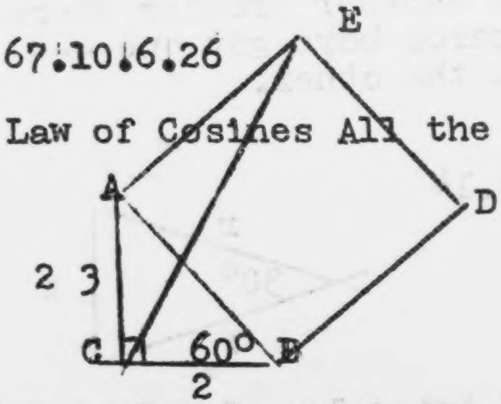
$\cos(D) = (36+25-x^2)/(60)$.
But since angle B is the supplement of angle D, $\cos(B) = -\cos(D)$. Therefore,
$(24)(x^2-61) = (60)(25-x^2)$.
And $x^2 = 247/7$. Good luck.

67.9.6.39

$(7x-1)/(x^2-x-6) = (7x-1)/(x-3)\cdot(x+2) = A/(x-3) + B/(x+2) = (Ax+2A + Bx-3b)/((x-3)(x+2)) = ((A+B)x + (2A-3B))/((x-3)(x+2))$;
Therefore, $A+B = 7$, $2A-3B = -1$.

67.10.6.26

Law of Cosines All the Way:



$AE = AB = 4$; 30-60-90 right triangle. Angle $CAE = 120°$.
Therefore,
$CE^2 = 12 + 16 - 2(2\sqrt{3})(4)(-1/2)$
QED.

67.11.5.A

$((3)(4)(5)(7)(2)) + 2$

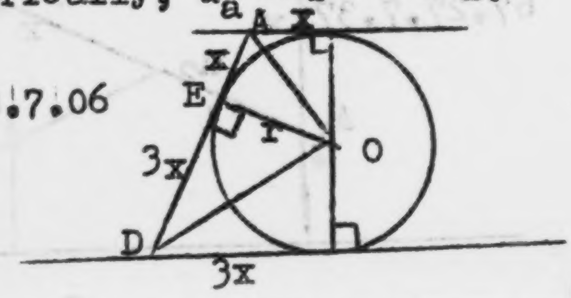
67.12.6.18

$u_0 = 3$; $u_1 = 7$; $u_2 = 15$;

$u_3 = 31$; $u_4 = 63$; etc. By

observation, $2^n - 1$. More specifically, $u_a = 2^{a+2} - 1$.

67.13.7.06

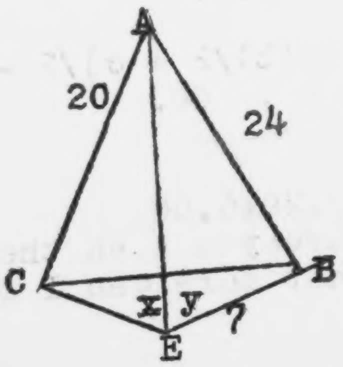


Area $ABCD = (1/2)(2x+6x)h =$
$(4x)(2r) = 8rx$
Area of $O = \pi r^2$. Therefore
$O/ABCD = (\pi/8)(r/x)$
Angle AOD is a right angle;
And OE is perpendicular to AD;
Therefore, $r^2 = (x)(3x)$.
Substituting in the ratio yields $\pi\sqrt{3/64}$.

67.14.7.28

DS = Dwarf Steps; GS = Giant Step
8/3 DS / GS (Time-wise)
11/2 DS per GS (Distance-wise)
If x = number of giant steps, then $(8x/3) + 85 = x(11/2)$,
That is, dwarf steps = dwarf steps.

67.15.6.07



My Kind of Meat!
$AE = 25$; 7-24-25 right triangle
$CE = 15$; 3-4-5 right triangle
$\cos(x) = 3/5$; $\sin(x) = 4/5$
$\cos(y) = 7/25$; $\sin(y) = 24/25$
$BC^2 = 225+49 - 2(15)(7)\cos(x+y)$
And $\cos(x+y) = -75/125$.
Compute.

67.16.6.76

64 man-hours = 1 unit of work.
$(36/64)(6) = 27/8$ units of work.

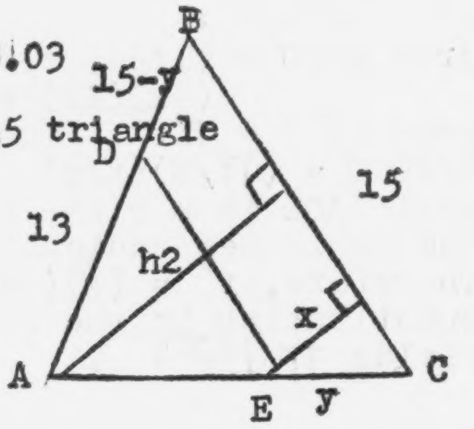
67.17.6.03

$$\sqrt[3]{6x+7} - \sqrt[3]{6x-7} = 2$$
$$A - B = 2$$

Cube it:
$6x+7-3A^2B+3AB^2-(6x-7) = 8$
$14 - 3(AB)(A*B) = 8$. But $A-B=2$
And $(AB)(A-B) = 2$; $AB = 1$.
Cube it and go.

67.18.8.03

13-14-15 triangle



Area of ABC = 1/2(14)(12) = 84
Area of ABC = 1/2(15)(h2) = 84
h2 = 56/5
From similar triangles,
1) x/h2 = y/14; x = 4y/5
2) (15-y)/13 = y/14; y = 70/9
Go.

67.19.6.76

((a+b)/2 + c)/2 - (a+((b+c)/2)/2 = 6. GO.

67.20.6.00
Have you seen the answer?
What more can I say?

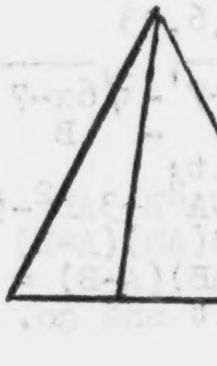
67.21.4.47

Greatest common multiple of 36: 36, 72, 108, 144, 180, 216. Observation: (3)(72) = 2(108). And 72 ≠ k(108) for k an integer.

67.22.6.67

Price per dozen eggs = x;
Price per egg (original) = x/12
Price per egg (new) = x/13 ;
Therefore, x-4 = 12(x/13).

67.23.6.22



$\cos(B) = \cos(C)$
$\cos(B) = (289+x^2-225)/(34x)$
$\cos(C) = (289+(2x)^2-225)/(68x)$
Therefore, $2x^2 = 64$; BC = 3x.

67.24.7.02

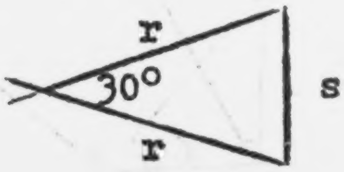
$z^x = y^{2x}$; $z = y^2$ only if $x \neq 0$

$2^z = 2(4^x)$; $z = 2x + 1$. Taken together,

$$x = (y^2-1)/2$$

Therefore,
$x+y+z = 16$ becomes $((y^2-1)/2) + y + y^2 = 16$. $y = 3, -11/3$.
BUT What if $x = 0$; then $z = 1$ and $y = 15$. It checks all around.
This is another problem where the fakeroo boys got one --- but not the other.

67.25.6.14



Area of triangle = 192/12 = 16
$= (1/2)r \cdot r \cdot \sin(30^\circ)$
Therefore, r = 8.
Law of Cosines:
$s^2 = 64 + 64 - 2(64)(\quad 3/2)$
QED

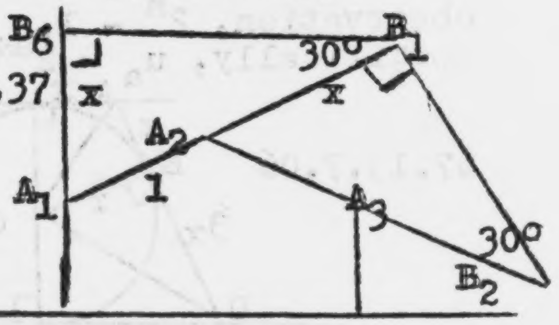
67.26.5.71

Units digit observation:

$1^5 = 1$; $2^5 = 3\underline{2}$; $3^5 = 24\underline{3}$;

$4^5 = \ldots\underline{4}$; $5^5 = \ldots\underline{5}$; etc.
Therefore, number ends in 1.
$10^5 = 100{,}000$; $20^5 = (32)(10^5) = 3{,}200{,}000$. Answer: 21.

67.27.7.37



Hexagon within a hexagon!
$x = (1/2)(1+x)$; $x = 1$.

67.28.8.10

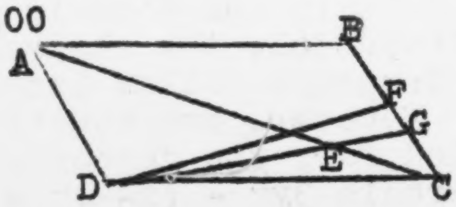
121, 144, 169, 196
121, 441, 961, 691

Since $144 = 12^2$, supposedly 441 = 144 in some other base. Therefore, $x^2+4x+4 = 441$, where x represents the new base.
$x^2 + 4x - 437 = 0$
$(x-19)(x+23) = 0$
Likewise,
$x^2 + 6x + 9 = 961$
$x^2 + 6x - 952 = 0$
$(x-28)(x+34) = 0$.
(Hey, did you take a close look at those factors with respect to the perfect square in question???)

67.29.6.07

$a^2 = c^2-b^2 = (c+b)(c-b)$
$a = 105$; $a^2 = (105)^2 = (3)(5)(7)(3)(5)(7)$
Therefore, $(c+b)(c-b)$ = that product.
For primitive triplets, the choice of factors would be:
$(1)(105)^2$; $(9)(35)^2$; $(25)(21)^2$; $(49)(15)^2$.

67.30.8.00



DE=9
FG=2
DF=12
AC= x
AE/EC = 3/1. Since triangles ADE and CGE are similar, EG = 3, DG = 12, and DGF is isosceles. GC = (1/3)AD
GC + 2 = (1/2)BC = (1/2)AD
Therefore, 3GC = 2GC + 4;
GC = 4.
And now, Law of Cosines:
From triangle DFG:
-cos(EGC) = cos(FGD) = 1/12
(immediately known because of isosceles triangle). From triangle EGC:
$EC^2 = 9+16 - 2(12)(-1/12) = 27$
$AC^2 = 16(EC)^2 = 432$.

1967 - 1968

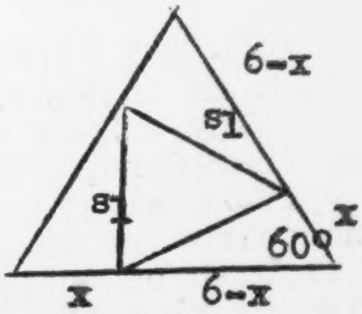
68.1.5.51

$44^2 = 1936$; $1936 - 44 = 1888$
$43^2 = 1849$; $1849 - 43 = 1806$
$42^2 = 1764$; forget it.

68.2.7.00

Working from the second equation only!!! $(xy-1)(xy-1) = (25/6)xy$; $(xy-1)^2 - (25/6)xy = 0$
Expanding, $6(xy)^2 - 37xy + 6 = 0$;
Therefore, $xy = 1/6$ or 6; and $y = 1/(6x)$ or $6/x$.
Taking each of these and substituting into the first equation will give in both cases quartic equations with no cubic or 1st degree terms. These quartics both factor, giving four solutions for each x-y relationship given above. Take it away.

68.3.7.04

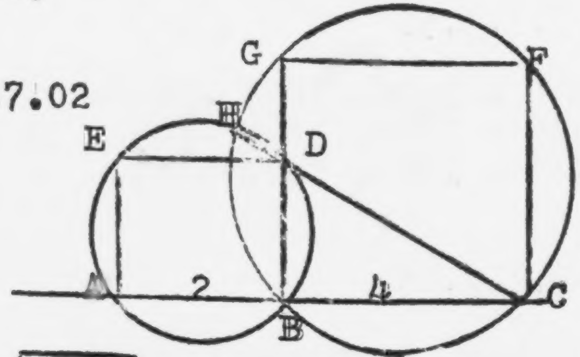


Area = $(s^2\sqrt{3})/4 = 9\sqrt{3}$,
Area/2 = $(9/2)\sqrt{3} = (s_1^2\sqrt{3})/4$
Therefore, $s_1 = 3\sqrt{2}$.
Law of cosines:
$(3\sqrt{2})^2 = x^2 + (6-x)^2 - 2x(6-x)(1/2)$
This should work into
$x^2 - 6x + 6 = 0$. Arithmetic.

68.4.5.16

1) $((a+b)/2)^2 - (\sqrt{ab})^2 = 25$; $(a^2+2ab+b^2)/4 - (4ab)/4 = 25$; $(a-b)^2 = 100$
2) $(a^2+b^2)/2 - (a+b)^2/4 = ?$
Substituting from 1) the value $a = b+10$ gives the desired answer.

68.5.7.02



$DC = \sqrt{4^2+2^2}$. Since HDC is a straight line (?), make use of the "Product of external segment of secant and entire secant is a constant"; use ABC and HDC as the secants to circle circumscribing square ABDE. Then $(4)(6) = (\sqrt{20})(\sqrt{q})$; $576 = 20q$ $\sqrt{q} = (1/5)\sqrt{k}$; $k = 25\,q$. For a simpler approach, try coordinate geometry.

68.6.6.47

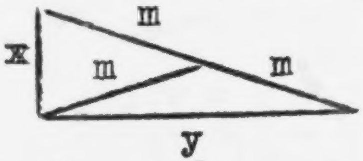
$x = 20 - 2y$; $13(20-2y) + 11y = 17k$, k an integer. Therefore $17k + 15y = 260$; $15y = 5$, mod 17 And therefore, $y = 6+17k$, $x = 20-2(6+17k)$. $k = 0$ is the only case where (x,y) will both be positive: therefore $(x,y) = (8,6)$.

68.7.6.19

First observe that $x \geq -1$ and $x \leq 12$ must at least be so; i.e., $-1 \leq x \leq 12$. However, the algebra adds additional restrictions.
$\sqrt{12-x} < 1+\sqrt{1+x}$; squaring both sides one gets $12-x < 2+x+2\sqrt{1+x}$ or $5-x < \sqrt{1+x}$. Therefore, $25 - 10x + x^2 < 1+x$; $x^2+11x-24 < 0$. The solution set of this inequality is $3 < x < 8$. The more restrictive of both cases therefore is $3 < x \leq 12$.

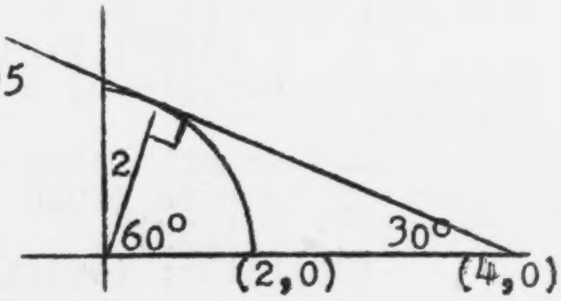
68.8.6.02

Note: The length of the median drawn to the hypotenuse of a right triangle is equal to one-half the hypotenuse. Therefore, from the figure,



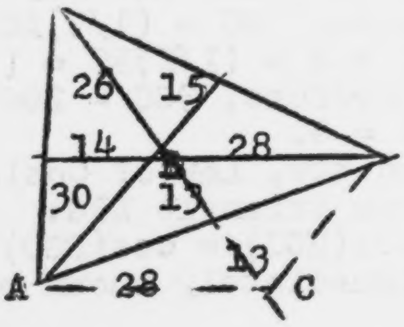
$x^2+y^2 = (2m)^2 = 4m^2$. But $m^2 = xy$ is given. Therefore, $x^2+y^2 = 4xy$; and $x^2+2xy+y^2 = 6xy$, (which is the whole ball game because) $(x+y)^2 = 6xy$; $x+y = \sqrt{6}\sqrt{xy}$; just set up the requested ratio, substitute in for $x+y$, and you're home free.

68.9.7.05



Merely recognize that the triangle in question is a 30-60-90 job; the slopes of the tangents follow immediately. However, one equation is called for; the degenerate hyperbola $3y^2 - (x-4)^2 = 0$ is the way to describe the two lines.

68.10.7.01



Find the area of ABC (using Heron's formula), take 1/2 of it and multiply by 6. This process can be generalized to obtain a very unique theorem.

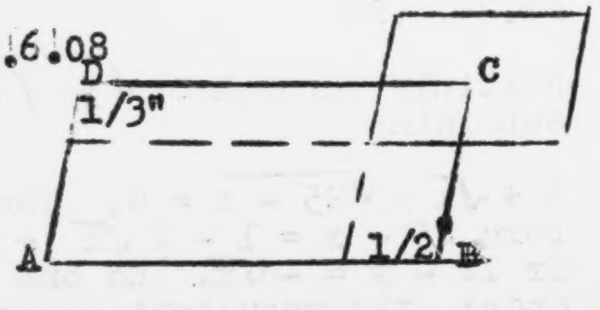
68.11.6.03

Each entry in (x,y,z) must be 0,±1,±2 This gives 5^3 = 125 points. Each entry ±2 (2^3 = 8) of the above points are outside the sphere. Each entry of ±(2,2,1) is on the sphere, and there are (8)(3) or 24 such points. Therefore, in the interior of the sphere there are 125 - 8 - 24 = 93 points.

68.12.6.43

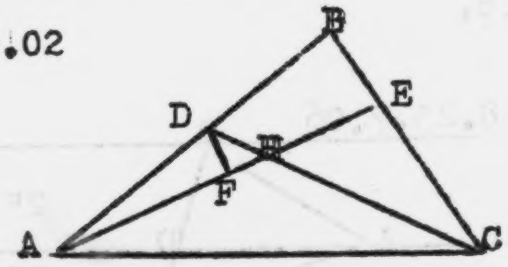
$$\begin{array}{rrrrr} 1 & 9 & 22 & 13 & -2 \\ & -2 & -14 & -16 & \\ \hline 1 & 7 & 8 & \underline{-3} & \\ & -2 & -10 & & \\ \hline 1 & 5 & \underline{-2} & & \\ & -2 & & & \\ \hline \underline{1} & \underline{3} & & & \end{array}$$

68.13.6.08



Area = absin(C) = 1(2/3)sin(60°)

68.14.9.02



Draw DF parallel to BC. Then DF/BE = 1/2; BE/EC = 1/2; DF/EC = 1/4. So DM/MC = 1/4 or MC = (4/5)CD = (4/5)(35/2). AF/AE = 1/2 and FM/FE = 1/4; thus FM = (1/5)FE = (1/10)AE; but AF = (1/2)AE. Therefore AM =(3/5)AE = (3/5)(65/3) = 13. Now, triangle AMC is the infamous 13-14-15 triangle, whose area is 84. Therefore, MEC/AMC = 2/3 or MEC = 56. Likewise, AEC/ABC = 2/3. ABC = 210.
Try coordinate geometry: it can be done in three steps.

68.15.5.22

n=1; sum = 1, first term = 1
n=2; sum = 10, second term = 9
n=3; sum = 27, third term = 17
Arithmetic sequence: therefore rth term = 8r-7.

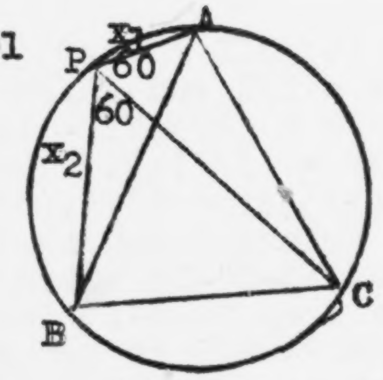
68.16.6.51

Most successful results are obtained by Fakeroo: The Devereaux-Duffrin approach goes like this, however. 11k = 3a+2 = 5b+1 = 8c+5; i.e., (11k-2)/3, (11k-1)/5, (11k-5)/8 must be integers. Or, ((9k-3)+(2k+1))/3, ((k-1)+10k)/5, ((8k-8)+(3k+3))/8 must be integers. Or (2k+1)/3, (k-1)/5, (k+1)/8 must be integers. Looking at the last two relationships, k = 31 is rather apparent. Therefore, (11)(31) = 341 does the job.

68.17.7.25

Don't get fancy, just blast away. Multiply the left side together: the zero solution is immediately apparent. Synthetic Substitution gives the second root; good ole quadratic formula does the rest.

68.18.8.01



From either triangle APC or triangle BPC, and the Law of Cosines:
$361 = x_i^2 + 441 - 2(21)(x_i)(1/2)$

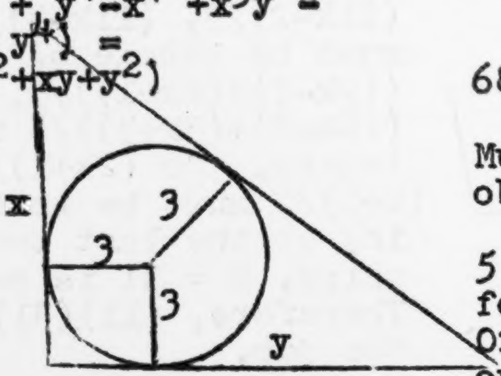
for $i = 1,2$.
Solving the quadratic yields $x_1 = 5, 16$.
Area of triangle BPA (Hero's Formula): $s = (5+16+19)/2 = 20$
Area $= 20\sqrt{3}$.
Area of triangle ABC $=$ $(361\sqrt{3})/4$. ACPB $= 441\sqrt{3}/4$

68.19.6.19

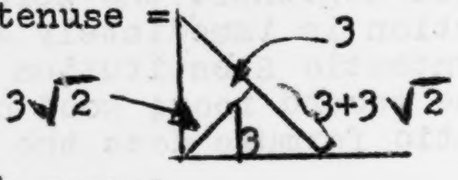
$(x+y)^5 - (x^5+y^5) =$

$(x+y)((x+y)^4 - (x^4-x^3y+x^2y^2-xy^3+y^4)) = (x+y)(x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 - x^4 + x^3y - x^2y^2 + xy^3 - y^4) =$
$(x+y)(5xy)(x^2+xy+y^2)$

68.20.6.06



x, y must be greater than 6.
First possible integer is 7.
Therefore, 7-24-25
8-15-17
9-12-15 Reject!
And that's it. Take 45° case.
Min hypotenuse =



$2(3+3\sqrt{2}) \approx 14.4$.
Therefore, 15 is the smallest.

68.21.6.35

Start dividing; since $(6)(17) = 102$, keep dividing until a remainder of 10 appears.

68.22.7.05

Simplify and add, getting:
1) + 2); $(x+y)^2 = 0$, $x+y = 0$
1) + 3): $(z+x)^2 = 9$, $x+z = \pm 3$
2) + 3): $(y+z)^2 = 1$, $y+z = \pm 1$
Pair the first relationship with the other 4 relations taken two at a time; this will give the 4 solution sets.

68.23.6.11

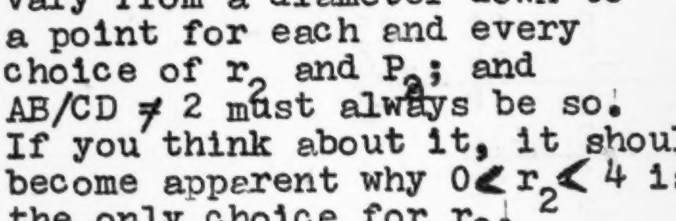
This problem is a "look-at"; take all possible variations. But remember that AB is to vary from a diameter down to a point for each and every choice of r_2 and P_2; and $AB/CD \neq 2$ must always be so. If you think about it, it should become apparent why $0 < r_2 < 4$ is the only choice for r_2.

68.24.6.22

Multiply through by $\sqrt{5+\sqrt{x}}$, obtaining

$5 + \sqrt{x} - \sqrt{25 - x} = 6$. Therefore $25 - x = 1 - 2\sqrt{x} + x$ Or $12 - x = -\sqrt{x}$. Go one more once: the resultant quadratic has roots of 16 and 9. But the 9 is extraneous --- drop it.

68.25.7.06



AB = BC = AC = x; DC = (1/3)x

Law of Cosines in ADC:
$AD^2 = x^2+(x/3)^2-2x(x/3)(1/2)$
Therefore, $AD = (\sqrt{7}/3)x$.
Area of ABD $= (2)(\sqrt{7}/3)(x)(1/2)$
Area of ADC $= (\sqrt{7}/3)(x)(1/2)$
Area of ABC $= (3/2)(\sqrt{7}/3)(x)$
$= (\sqrt{7}/2)(x)$.
Also, ABC $= (\sqrt{3}/4)x^2$.
Solve for x and approximate.

68.26.5.26

$2^{38}=(2^7)^5(2^3)= (1)(8)$, mod 127.

68.27.7.05

For first meeting, in time t1, A travels x-72 yards, while B travels 720 yards. Therefore $\text{Rate}_A = (x-720)/t1$ and $\text{Rate}_B = 720/t1$; $R_A/R_B= (x-720)$ divided by 720.
For the second meeting, after time t2, A travels a total distance of $x+(x-400) = 2x-400$ yards, while B travels a total distance of $x+400$ yards. Therefore, $R_A = (2x-400)/t2$; $R_B = (x+400)/t2$; $R_A/R_B = (2x-400)/(x+400)$.
If the two ratios are equated, and solved, the answer pops out.

68.28.6.30

From the way this problem is presented, one should be able to assume that the area in question is an invariant; i.e., not dependent on where one places the perpendicular. Therefore, let the perpendicular = 8 pass through the center of the semi-circle.
Then $A_P = \pi(8)^2/2 = 32\pi$
$A_R = A_S = \pi(4)^2/2 = 8\pi$
$P - (R+S) = 16\pi$.

68.29.6.12

Multiplying, one gets:

$(x+2y+2z) + (x+y+2z)\sqrt[3]{2} + (x+y+z)\sqrt[3]{4} = 1 + 0\sqrt[3]{2} + 0\sqrt[3]{4}$
Equating corresponding coefficients : $x+2y+2z = 1$; $x+y+2z = 0$; $x+y+z = 0$. Solve.

68.30.3.23

Must know: $m = h/2 = 4\sqrt{2}$;
$m^2 = xy$; Area $= xy/(2)$
QED

1968 - 1969

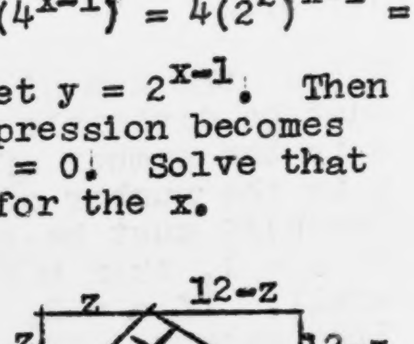
69.1.4.41

$3^{3^3} \neq (3^3)^3$. Merely look in the denominator if in doubt.

69.2.6.04

Since $4^x = 4(4^{x-1}) = 4(2^2)^{x-1} = 4(2^{x-1})^2$. Let $y = 2^{x-1}$. Then the given expression becomes $4y^2 -17y + 4 = 0$. Solve that and go back for the x.

69.3.7.08



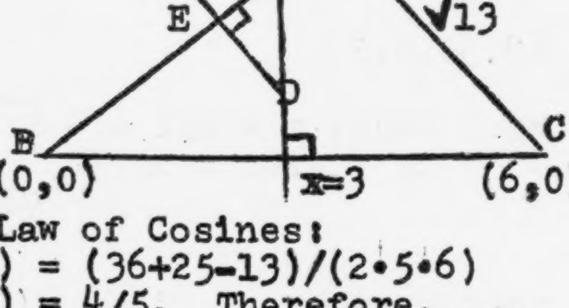
$xy = (1/3)(144) = 48$
$x^2 = 2z^2$; $x = \sqrt{2}z$
$y^2 = 2(12-z)^2$
Therefore, $x^2y^2 = 4z^2(12-z)^2$
$xy = 2z(12-z) = 48$. Solve for z.

69.4.7.32

Most Mathletes try to fake this directly: sheer madness. First eliminate one of the unknowns and then fake it!
$7A + 6B + 4C = 59$;
$4A + 5B + 6C = 48$. Eliminate the C term and obtain $13A+8B=81$.

Obviously, $A=5$ and $B=2$; substituting, $C = 3$.

69.5.7.02



From Law of Cosines:
$\cos(B) = (36+25-13)/(2\cdot 5\cdot 6)$
$\cos(B) = 4/5$. Therefore, the coordinates of A are $(4,3)$; slope of line $AB = 3/4$; slope of line $DE = -4/3$. $E(2,3/2)$. Equation of line ED: $4x+3y-25/2 = 0$. y coordinate of D is when $x = 3$. Therefore, $D(3,1/6)$
$BD = r = \sqrt{9 + 1/36}$.

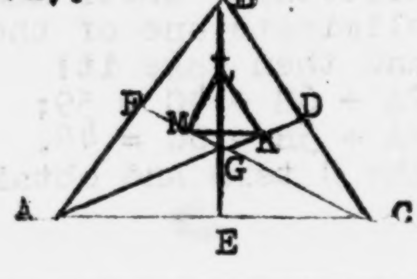
69.6.7.28

Rule: the sum of the digits must be divisible by 9. If a is the number of 3's and b is the number of 2's, then $(3a+2b)/9$ must be an integer. If $a = 1$, then $b = 3$; too small. If $a = 2$, then $b = 3/2$; reject. If $a = 3$, then $b=0$; no good. If $a = 4$, then $b = 3$; OK.

69.7.4.06

The meaningful form of the division algorithm should be used: i.e., "When a is divided by b, the quotient is c and the remainder is d" should be translated into $a = bc + d$. Similarly, the second statement means $c = eq + f$. Therefore, $a = b(eq+f) + d = (beq) + (bf+d)$.

69.8.4.21



$LM = (1/2)BF = (1/4)AB$
$LK = (1/2)BD = (1/4)BC$
Since angle MLK is congruent to angle ABC, triangles LMK and ABC are similar. Therefore the ratio of the areas is 1/16.

69.9.7.14

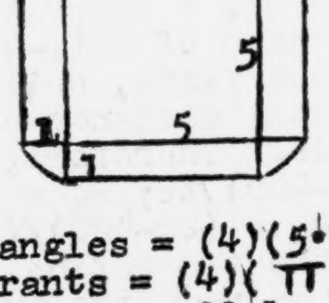
$$f(n+1) = ((1+\sqrt{5})/2)((1+\sqrt{5})/2)^n + ((1-\sqrt{5})/2)((1-\sqrt{5})/2)^n$$

Likewise,

$$f(n-1) = ((1+\sqrt{5})/2)^{-1}((1+\sqrt{5})/2)^n + ((1-\sqrt{5})/2)^{-1}((1-\sqrt{5})/2)^n$$

In forming the difference, $f(n+1) - f(n-1)$, note the common factor of $f(n)$; factoring it out and simplifying the remaining terms gives the desired result.

69.10.5.36



Area of rectangles $= (4)(5\cdot 1)$
Area of quadrants $= (4)(\pi/4)$
Sum of those areas $= 23.1$

69.11.6.38

Repeated divisions seem to be the most popular fashion. However, since $2002 = (2)(1001) = (2)(11)(91)$ and $182 = (2)(91)$, and since $1001 - 1 = 10^3$, so therefore $(10^3)^3 = 10^9 = -1$, modulo 91. Therefore $10^9 + 1 = 0$, mod 91. Throw the 2 back in.

69.12.6.27

$\log_{25}x + \log_{x^2}5 = 1$. In a nutshell, use change of base theorem, clear the denominator and look for the quadratic. You should end up with:

$(\log_5 x)^2 - 2\log_5 x + 1 = 0.$

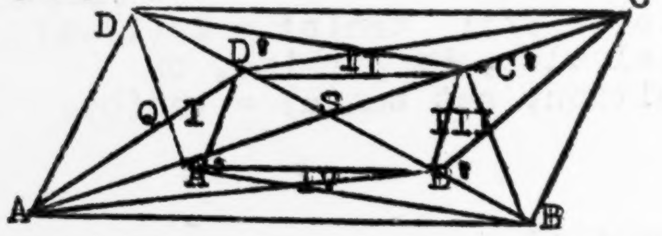
69.13.5.34

By fakeroo, n-1 n-gons. Technically, keeping 1 point fixed, the other point can be any one of n-1 vertices.

69.14.6.24

P(2x) = 2P(x) implies that the polynomial is of first degree. Only possibility. Since $P(x) = a_1 x$ and $P(3) = 12$, so $a_1 = 4$.

69.15.6.22



Area of ABCD = 48
AREA of SDC = (1/4)48 = 12
Area of SD'C' = $(1/2)^2(12) = 3$
Area of A'B'C'D' = 4(3) = 12
Triangle AQD = (1/3)ASD = 4
Triangles AQD and A'QD' are similar.
Area of triangle A'QD' = $(1/2)^2$(triangle AQD) = 1
Likewise, areas of triangles II,III,IV each equal 1.
Area of region = 12+4

69.16.6.01

The number of divisors of a number n is equal to $(k_1+1)(k_2+1)\ldots(k_n+1)$, where the k_i's of these numbers are found in the relationship $n = p_1^{k_1} \cdot p_2^{k_2} \ldots p_n^{k_n}$,

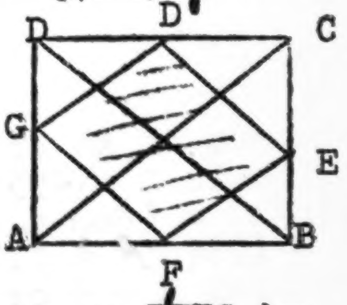
where the p_i are primes. Therefore, $15 = (k_1+1)(k_2+1)$ or $15 = (3)(5)$; $k_1 = 2$, $k_2 = 4$. Thus for the smallest number with 15 divisors, $n = p_1^2 \cdot p_2^4$, or $n = p_1^{14}$; let $p_1 = 5$, $p_2 = 3$. Answer: 2025

An intuitive approach: since there must be 15 divisors, there must be 7 pairs of factors and one factor must match with itself; therefore the number n in question must be a perfect square! Start faking with factors of 3's and 5's. 2025 is indeed 45^2.

69.17.6.50

12A + 7B = 1; 7A + 14B = 1
A = 1/17, B = 5/119.

69.18.7.54

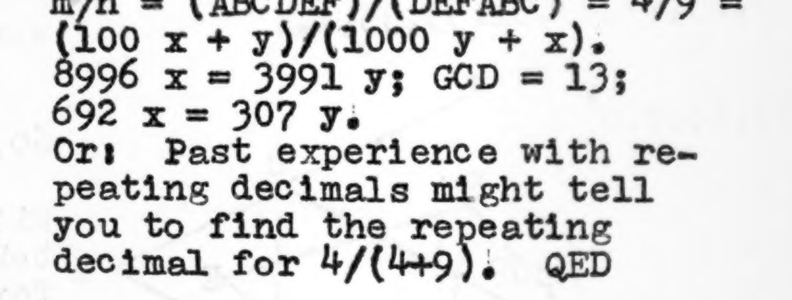


Take Frinstances: DEFG is square whose area must be found.

69.19.6.01

Algebraically:
m/n = (ABCDEF)/(DEFABC) = 4/9 = (100 x + y)/(1000 y + x).
8996 x = 3991 y; GCD = 13;
692 x = 307 y.
Or: Past experience with repeating decimals might tell you to find the repeating decimal for 4/(4+9). QED

69.20.7.05



Note: segment OFO' = 12+3= 15
3-4-5 triangles are: EOO', POB

and PFG.
Triangle EOO' is 9-12-15
Triangle POB is 12-16-20
Triangle PFG is 6-8-10
Even triangle APO' is 3-4-5
Area of PGH = (6)(8) = 48

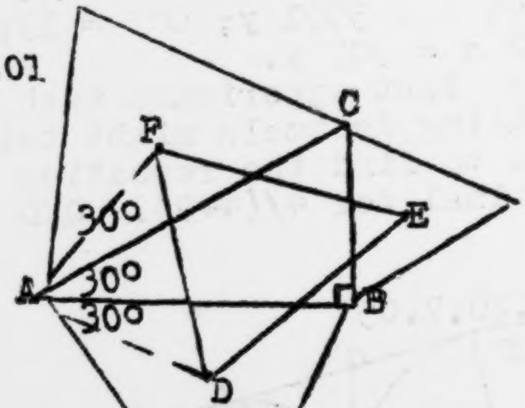
69.21.7.A

Let $5x+2y = ap$
$7x+5y = bp$
Eliminating the y term,
$11x = (5a-2b)p$
Eliminating the x terms:
$11y = (5b-7a)p$
Since p,x,y are relatively prime, p = 11.

69.22.7.A

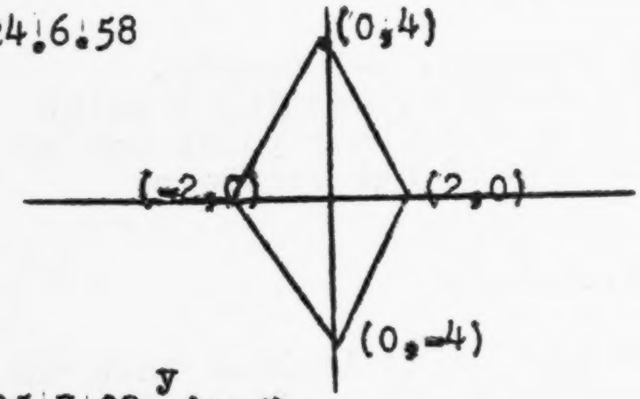
Since the sum of an arithmetic progression is $S = 455 = (a_1+a_n)(n/2) = ((a_1+a_n)/2)(n) = (5)(91) = (7)(65) = (13)(35)$, so therefore, n = 5 or 7 or 13 and 91, 65, and 35 are the corresponding middle terms (average term) of the arithmetic sequences. The possible sequences are therefore
87,89,91,...
59,61,63,65,...
23,25,27,29,31,33,35,...

69.23.7.01

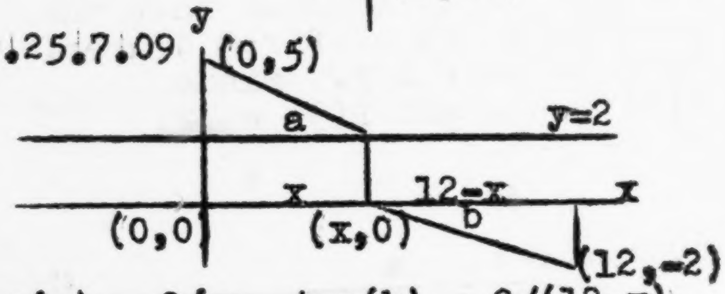


$AB = 12$, $BC =$ [illegible]$\sqrt{3}$, $AC = 8\sqrt{3}$
$AF = (2/3)(1/2)(AC)(\sqrt{3}) = 8$
$AD = (2/3)(1/2)(AB)(\sqrt{3}) = 4\sqrt{3}$
$FD^2 = 64 + 48 = 112$
Since triangle FDE is equilateral, area of triangle FDE is $(112\sqrt{3})/4$.
Note: Use the Law of Cosines on triangles DBE and CFE to verify that triangle FDE is indeed equilateral.

69.24.6.58



69.25.7.09



tan(a) = 3/x; tan(b) = 2/(12-x)
By calculus, by physics, by intuition, set tan(a) = tan(b).
Go.

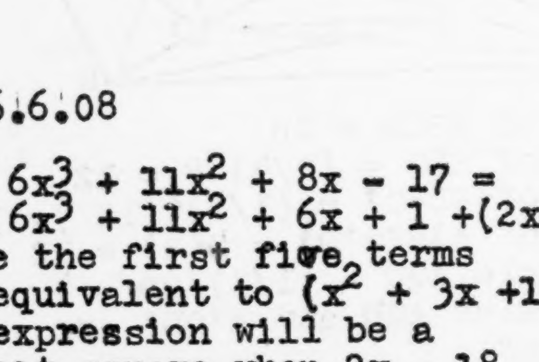
69.26.6.08

$x^4 + 6x^3 + 11x^2 + 8x - 17 =$
$x^4 + 6x^3 + 11x^2 + 6x + 1 + (2x-18)$
Since the first five terms are equivalent to $(x^2 + 3x + 1)^2$, the expression will be a perfect square when $2x - 18$ vanishes.

69.27.7.67

With four conditions taken three at a time, solve for x, y, and z and substitute.

69.28.7.07



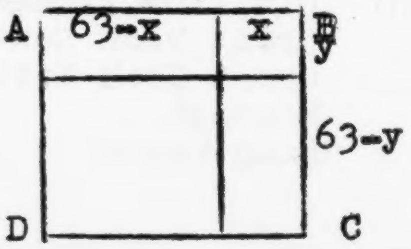
Angle BOC = 120°; angle BOP = 60°. BC = $6\sqrt{3}$; OC = OB = 6.
OD = 3, DB = $3\sqrt{3}$.
AREA Sector OCB = $(1/3)\pi(36)$
Area triangle OBC = twice the area of ODB = $(2)(9\sqrt{3}/2)$
Shaded area = $12\pi - 9\sqrt{3}$
Intersection = $24\pi - 18\sqrt{3}$
Union = $72\pi - (24\pi - 18\sqrt{3})$.

69.29.6.05

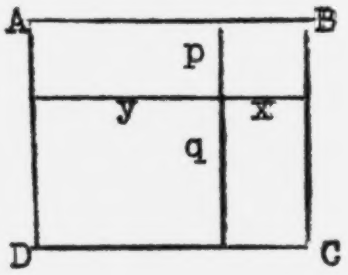
$f(-1) = 0 = a + b|k-1|$;
$|k-1| = -b/a$
$f(2) = 0 = 2a + b\,|k+2|$;
$(1/2)\,|k+2| = -b/a$
And
$2\,|k-1| = |k+2|$. Using alternate definition of $|x|$, i.e., $|x| = \sqrt{x^2}$, and $|x|^2 = x^2$,
$4k^2 - 8k + 4 \div =$
$k^2 + 4k + 4$
Solve.

69.30.6.20

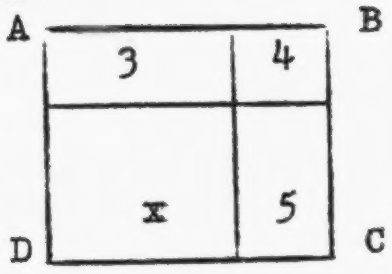
Easy way: Two unknowns, x and y, (as in the diagram below), and solve.



Easier way: (See diagram below). $x/y = 4/3$; therefore
$x = (4/7)(63) = 36$
$y = (3/7)(63 = 27$
Also, $p/q = 4/5$;
And $q =(5/9)(63) = 35$
$R_D = (35)(27) = 945$



Easiest Way:



$3/4 = x/5$; $x = 15/4$
Ratios: 3:4:5:15/4
Actual values : $3a, 4a, 5a, (15a/4)$
$3a + 4a + 5a + 15a/4 = 63a/4 = 63^2$
$a = 4(63)$; $15a/4 = 945$.

APPENDIX A: Some History of Success in the NCIML

1955 - 56

First year; five meets, 18 schools, and only 187 participants.

First	Long Beach	61 pts
	Massapequa	61
	Lawrence	60
	Mineola	58
	Hewlett	57

1956 - 57

Still five meets; first year for Mr. Harry Sitomer authors the problems. 23 schools, 236 students participate.

First	Hewlett	64 pts
	Lawrence	59
	Baldwin	55
	Sewanhaka	52
	Wantagh	48

1957 - 58

Five meets, 33 teams, and 329 participants.

First	West Hempstead	59 pts
	Farmingdale	52
	East Meadow	51
	Long Beach	49
	Wheatley	49

1958 - 59

Five meets, 41 teams, and 426 students participate.

First	Hewlett	44 pts
	Great Neck South	43
	Long Beach	43
	Lynbrook	40
	Carey	39

1959 - 60

Five meets, 46 teams and 481 participants.

First	Long Beach	63 pts
	Great Neck North	61
	Valley Stream North	61
	Sewanhaka	58
	Port Washington	58

1960 - 61

Six meets started; 48 teams and 481 students participate.

First	Great Neck North	71 pts
	Great Neck South	51
	Elmont Memorial	51
	Valley Stream N.	50
	Chaminade	48

1961- 62

Six meets, 49 teams and 536 participants.

First	Division Avenue	67 pts
	Great Neck South	62
	Great Neck North	57
	Wantagh	57
	Long Beach	52

1962 - 63

Six meets of course; 51 schools and 534 participants.

First	Great Neck South	83 pts
	Long Beach	78
	Great Neck North	76
	Elmont Memorial	71
	Herricks	71

1963 - 64

Six meets with 596 participants from 52 different schools.

First	Great Neck South	92 pts
	Great Neck North	82
	Long Beach	77
	Valley Stream N.	68
	MacArthur	66

1964 - 65

Six meets with 568 participants from 51 different schools.

First	Great Neck South	67 pts
	Valley Stream N.	63
	Wantagh	63
	Mepham	60
	West Hempstead	56

1965 - 66

Six meets with 620 students from 55 schools.

First	Great Neck South	69 pts
	Valley Stream N.	66
	Wantagh	57
	Mepham	53
	Hicksville	50

1966- 67

Six meets with 639 students from 59 schools.

First	Great Neck South	75
	Syosset	74
	Valley Stream N.	66
	Plainview - OB	63
	Great Neck North	62

1967 - 68

Six meets, 655 students, 59 schools.

First	Great Neck South	64 pts
	Valley Stream N.	63
	Syosset	52
	Calhoun	48
	Baldwin	45

1968 - 69

Six meets, 645 students, 61 schools.

First	Port Washington	71 pts
	Valley Stream N.	69
	Syosset	66
	Plainview - OB	59
	Elmont Memorial	58

Top Mathletes Through the Years

Year	Mathlete and School	Points
1956	D. Friedman Long Beach J. Solomon Massapequa Bob Victor Farmingdale	17
1957	Bob Victor Farmingdale	17
1958	Paul Ermak Oceanside	19
1959	Steve Kessler Hewlett	15
1960	Mike Green Lawrence	21
1961	C. Gulizia Sewanhaka	21
1962	S. Schulman Division Avenue	21
1963	D. Oestreicher Great Neck South	21
1964	Ira David Great Neck South S. Schulman MacArthur	22
1965	W. Grossman Roslyn	21
1966	Dave Relson Great Neck South	21
1967	Dave Ellis Plainview - OB Doug Lublin Great Neck North Steve Stein Syosset	21
1968	Steve Chessin Great Neck South	19
1969	Mike Glantz Valley Stream North	24

APPENDIX B: PLAY-OFF PROBLEMS THROUGH THE YEARS

Note: There are no percentages on these problems as they were only attempted by a select few; in addition to that, most of the times given on these problems are much shorter than would normally be allowed. This is due to the nature of our Play-Offs; certain very sharp Mathletes had to be eliminated. For a few years, again some of these times have been mislaid; but you can imagine for yourself what they might have been.

1959 - 1960

PO.60.1.4

Suppose that all the integers from 1 to 1300 have been factored into prime integers. (1 is not a prime.) Which of these numbers has the largest number of factors ?

PO.60.2.5

The arms of a right triangle are 21 and 28 inches respectively. A square is drawn so that one of its angles coincides with the right angle of the triangle and a vertex of the square lies on the hypotenuse. How many inches are there in each side of the square ?

PO.60.3.5

If $4^x - 4^{x-1} = 48$, find the numerical value of x^{-x}.

PO.60.4.4

ABCD is a square. Equilateral triangle ABE is drawn within the square. DE, extended, meets CB in F. How many degrees are there in angle DFB ?

PO.60.5.5

In a group of 31 students a poll was taken as to whether they liked music, art or ballet. 2 liked only music. 12 liked only ballet. On the other hand 6 disliked only art, 1 disliked only ballet and 3 disliked only music. If there were only 3 who liked all three, how many liked only art ?

PO.60.6.6

Suppose that an object falls 16 t^2 feet in t seconds and that sound travels 1000 feet per second. Then when a pebble is dropped into a well the splash will be heard 10 seconds later. How deep should the well be, to the nearest 10 feet ?

PO.60.7.6

The rate of a stream is 2 miles per hour. A boat travels up this stream a certain distance and back. Its average rate for the entire trip is 16/3 miles per hour. Find the average rate of the boat in still water in miles per hour.

PO.60.8.6

Point E in BC of triangle ABC

is located so that CE is twice as long as BE. Through E a line is drawn parallel to AB intersecting AC in F and also a line through B parallel to AC intersecting AB in D. What fractional part of the area of triangle ABC is in triangle ADF ?

PO.60.9.5

If $(x+2)^2$ is greater than $5x+10$ but less than $2x+12$, what is the integral value of x ?

PO.60.10.7

In a given circle, chord AB = chord AC. Chord AD cuts BC in E. If AB measures 12 feet and AE measures 8 feet, how many feet are there in AD ?

1960 - 1961

PO.61.1.?

A and B race around a circular track with a circumference of 100 yards. If they start together, A passes B in 1 minute, 15 seconds. If B has a 10 second head start, A passes him in 50 seconds. Find B's rate in yards/second.

PO.61.2

The lines $y = 3x - 9$, $y = 16 - 2x$, and the x-axis form a triangle with k square units area. Find k.

PO.61.3

In how many different ways can 50 coins (5¢, 10¢, 25¢) total $5.00 ?

PO.61.4

Find the length, in inches, of the angle bisector drawn to the longer leg of a triangle with sides of 3, 4, and 5 inches.

PO.61.5

At a PTA affair attended by parents and children, the number of females is 2/3 the number of males; 1/2 the males are boys; 28 of the females are girls; the husbands of 1/3 of the mothers are present and the wives of 1/4 of the fathers are present. How many people attended the affair ?

PO.61.6

Solve for 2 values of x:

$$x^{\log_{10} x} = 1{,}000{,}000\,x \,.$$

PO.61.7

In square ABCD, AB = 2; E, F, G, and H are the midpoints of AB, BC, CD, and DA, respectively. If AF, BG, CH and DE are drawn, find area of the resulting square, in square units.

PO.61.8

In an infinite geometric progression, the limiting sum is 15. If all the terms are squared, the limiting sum is 45. Find the first 3 terms.

1961 - 1962

PO.62.1.?

Triangle ABC is equilateral. Point P is in base AB nearer to A. Two perpendiculars are drawn from P meeting AC in D and BC in E. If the altitude of triangle ABC is 10", find the number of inches in DP + PE.

PO.62.2

Find the number of inches in the side of a square if its area is equal to that of a triangle whose medians are 30", 30", and 48".

PO.62.3

Find in radical form (but not a radical within a radical) the positive square root of $6 + \sqrt{8} + \sqrt{12} + \sqrt{24}$.

PO.62.4

The parabola $y = ax^2 + bx + c$ passes through $A(0,-1)$, $B(1,-2)$, $C(2,1)$ and $D(3,y_D)$. Find the value of y_D.

PO.62.5

A causeway which rises at a uniform grade is supported by 47 upright T-shaped pillars equally spaced. If the shortest pillar is 3' in height and the tallest 31', what is the total height in feet of these pillars, excluding the first and the last ?

1962 - 1963

PO.63.1.5

As the officials were arranging the Fourth of July parade, they say that if the marchers were put in ranks of 3 abreast, 2 men were left over; if put 5 abreast, there would be 4 left over; 7 abreast, 6 left over; 11 abreast, 10 left over. What is the least number of marchers there could have been ?

PO.63.2.5

A man started for a walk when the hands of his watch were coincident between 3 and 4 o'clock. When he finished his walk, the hands were again coincident between 5 and 6 o'clock. What, exactly, is the least number of minutes that he could have walked ?

PO.63.3.4

A ball is dropped from a height of 20'. It rebounds half the distance on each bounce. What total distance, in feet, does it travel ?

PO.63.4.5

7 farmers agree to share a circular grindstone equally. If each farmer grinds his share in turn, the second farmer will reduce the diameter of the stone by this fractional part of the original diameter: $(\sqrt{a} - \sqrt{b})/\sqrt{c}$, where a, b, and c are the smallest possible positive integers. Find a+b+c.

PO.63.5.4

If $u_{n+1} = iu_n + 1$, where $u_1 = i + 1$, and $i = \sqrt{-1}$, find u_{27} .

1963 - 1964

NONE

1964 - 1965

PO.65.1.6

A circle of radius $\sqrt{1105}$ is drawn on a Cartesian plane, centered at the origin. How many lattice points does it pass through ? (A lattice point has integers for both coordinates.)

PO.65.2.7

Triangle ABC is constructed with sides AB, BC, and CA equal to 85, 13, and 84 respectively. Angle bisector BD is drawn, after which a line through B is drawn to meet AC extended at point F such that BD is perpendicular to BF. If the area of triangle BDF is 85k/84, find k. (Note: D is a point lying on AC.)

PO.65.3.6

The B_n of a very famous sequence are defined in the following recurrent fashion:

$(B+1)^k - B^k = 0$ for $k > 1$;

where the exponents "become" subscripts after the expansion is performed. Find B_4 .

PO.65.4.6

A trapezoid ABCD is drawn with AB parallel to CD. A line through point E, the midpoint of AD, is drawn perpendicular to side BC extended, meeting it in point F. Segments BE and CE are drawn and found to be 26 and 20 units long respectively. BF is 24 units long. The area of trapezoid is $(20)(12 - \sqrt{k})$ square units. Find k.

PO.65.5.5

A certain number between 200 and 400 is divided by 7, 11, and 13 yielding remainders of 2, 1, and 6 respectively. Find the number.

PO.65.6.3

Find all the points which could serve as the fourth vertex of a parallelogram whose other vertices are to be (2, -31/6) , (-1/12,13/12) , and (-4,77/6) .

PO.65.7.5

Titterton's Theorem asserts that rational numbers whose numerators and denominators consist of two digit numbers not ending in zero can be reduced by "cancellation" (in the strict sense) of the units digit of the numerator and the tens digit of the denominator whenever they are equal; for instance, 16/64 = 1/4. Find one other such rational number (numerator ≠ denominator) which would serve as an additional "proof" of this theorem.

1965 - 1966

NONE

1966 - 1967

PO.67.1.5

Triangle ABC has sides $AB = 21$, $BC = 17$, $AC = 10$. The bisector of angle C intersects AB at D. The area of triangle ADC is given by $k/9$. Give the numerical value of k.

PO.67.2.3

Give in simplest radical form (a radical within a radical is not acceptable) the principal square root of $9 + 4\sqrt{5}$.

PO.67.3.7

Find the integral value of x^2 if $\sqrt[3]{x+8} - 1 = \sqrt[3]{x-8}$

PO.67.4.3

The integer 6162 is the product of four distinct primes. Give the numerical value of the largest of these primes.

PO.67.5.7

The rear wheel of a velocipede makes 110 more revolutions in a mile than the front wheel. If the circumference of the rear wheel were increased 25 per cent, and that of the front wheel decreased 25 per cent, then the front wheel would revolve 88 more revolutions than the rear wheel in going a mile. Find the circumference of the front wheel.

1967 - 1968

NONE

1968 - 1969

PO.69.1.4

The harmonic mean of two numbers a and b is given by $(2ab)/(a+b)$. If the 1st and 3rd terms of a harmonic sequence are the numbers 2 and 2/3 respectively, find the sum of the first six terms of the sequence.

PO.69.2.4

The medians drawn from the acute angles of a right triangle are 10 units and 15 units. Find the length of the median drawn to the hypotenuse of the triangle.

PO.69.3.3

A circle has a radius r and an equilateral triangle is inscribed in the circle. A second circle is inscribed in the equilateral triangle, and a second equilateral triangle is inscribed in it. Assuming that this process is continued, give in terms of r the altitude of the (n+1)st inscribed equilateral triangle.

PO.63.5.4

If $u_{n+1} = iu_n + 1$, where $u_1 = i + 1$, and $i = \sqrt{-1}$, find u_{27} .

1963 - 1964

NONE

1964 - 1965

PO.65.1.6

A circle of radius $\sqrt{1105}$ is drawn on a Cartesian plane, centered at the origin. How many lattice points does it pass through ? (A lattice point has integers for both coordinates.)

PO.65.2.7

Triangle ABC is constructed with sides AB, BC, and CA equal to 85, 13, and 84 respectively. Angle bisector BD is drawn, after which a line through B is drawn to meet AC extended at point F such that BD is perpendicular to BF. If the area of triangle BDF is 85k/84, find k. (Note: D is a point lying on AC.)

PO.65.3.6

The B_n of a very famous sequence are defined in the following recurrent fashion:

$(B+1)^k - B^k = 0$ for $k > 1$;

where the exponents "become" subscripts after the expansion is performed. Find B_4 .

PO.65.4.6

A trapezoid ABCD is drawn with AB parallel to CD. A line through point E, the midpoint of AD, is drawn perpendicular to side BC extended, meeting it in point F. Segments BE and CE are drawn and found to be 26 and 20 units long respectively. BF is 24 units long. The area of trapezoid is $(20)(12 - \sqrt{k})$ square units. Find k.

PO.65.5.5

A certain number between 200 and 400 is divided by 7, 11, and 13 yielding remainders of 2, 1, and 6 respectively. Find the number.

PO.65.6.3

Find all the points which could serve as the fourth vertex of a parallelogram whose other vertices are to be (2, -31/6) , (-1/12,13/12) , and (-4,77/6) .

PO.65.7.5

Titterton's Theorem asserts that rational numbers whose numerators and denominators consist of two digit numbers not ending in zero can be reduced by "cancellation" (in the strict sense) of the units digit of the numerator and the tens digit of the denominator whenever they are equal; for instance, 16/64 = 1/4. Find one other such rational number (numerator $\neq$ denominator) which would serve as an additional "proof" of this theorem.

1965 - 1966

NONE

1966 - 1967

PO.67.1.5

Triangle ABC has sides AB = 21, BC = 17, AC = 10. The bisector of angle C intersects AB at D. The area of triangle ADC is given by k/9. Give the numerical value of k.

PO.67.2.3

Give in simplest radical form (a radical within a radical is not acceptable) the principal square root of $9 + 4\sqrt{5}$.

PO.67.3.7

Find the integral value of x^2 if $\sqrt[3]{x+8} -1 = \sqrt[3]{x-8}$

PO.67.4.3

The integer 6162 is the product of four distinct primes. Give the numerical value of the largest of these primes.

PO.67.5.7

The rear wheel of a velocipede makes 110 more revolutions in a mile than the front wheel. If the circumference of the rear wheel were increased 25 per cent, and that of the front wheel decreased 25 per cent, then the front wheel would revolve 88 more revolutions than the rear wheel in going a mile. Find the circumference of the front wheel.

1967 - 1968

NONE

1968 - 1969

PO.69.1.4

The harmonic mean of two numbers a and b is given by (2ab)/(a+b). If the 1st and 3rd terms of a harmonic sequence are the numbers 2 and 2/3 respectively, find the sum of the first six terms of the sequence.

PO.69.2.4

The medians drawn from the acute angles of a right triangle are 10 units and 15 units. Find the length of the median drawn to the hypotenuse of the triangle.

PO.69.3.3

A circle has a radius r and an equilateral triangle is inscribed in the circle. A second circle is inscribed in the equilateral triangle, and a second equilateral triangle is inscribed in it. Assuming that this process is continued, give in terms of r the altitude of the (n+1)st inscribed equilateral triangle.

PO.69.4.4

A primitive Pythagorean triple is a triple of positive integers (x,y,z) such that $x^2 + y^2 = z^2$ and x, y, and z have no common factors. List all such triples satisfying the condition $x+y+z = 10q$ where q is an integer between 0 and 10.

PO.69.5.5

In triangle ABC, AC = 6, BC = 10, and angle C = 120°. The bisector of angle C intersects AB at D. Find the area of triangle ACD.

PO.69. Sudden Death 1.3

Four equal circles are inscribed in a square of side 3 units in such a way that each circle is externally tangent to two other circles and at the same time tangent to two sides of the square. The area of the sections between the exterior of the circles and the sides of the square is found to be $(a/b)(4 - \pi)$. Give the numerical value of a + b.

PO.69. Sudden Death 2.3

In Short Corners N. J. a poll was taken of x people to determine their reading preferences. It was discovered that:
86 people read the New York Times
74 people read the Wall St. Journal
123 people read the Daily News
50 people read both the Times and the Journal
33 people read both the Journal and the News
23 people read both the Times and the News
17 people read all three newspapers. Find x.

NASSAU - SUFFOLK MEET PROBLEMS

Although this competition has been held for ten years, I have only been able to gather together the problems from the last four years.
It is of some interest that in these ten years, Nassau has been victorius 100% of the time. The trophy for the meet resides annually in that Nassau High School that has been first in our League.

1965 - 1966

NS.66.1

How many numbers of 4 different digits each can be formed using the digits 1, 2, 3, 4 so that each number is exactly divisible by 11 ?

NS.66.2

Given: Triangle ABC; DE is a line parallel to base BC, cutting AB in D and AC in E. BC = b units; the area of triangle ADE is 2/3 the area of triangle ABC. Find the length of DE in terms of b.

NS.66.3

Given the sequence:
77, 7777, 777777, 77777777, ...
Find the nth term (not 2n 7's).

NS.66.4

Write the equation of a circle whose center is at the origin and which is tangent to the line $4x + 3y = 12$.

NS.66.5

A man buys 2 cars ans sells them for $2,400 each. On one, he gains 20% of the cost and on the other, he loses 20% of the cost. Did he gain, lose, or break even and how much ?

1966 - 1967

NS.67.1.5

A four digit number, 2abc, is multiplied by 4 and the result is the original four digit number with the digits reversed; i.e., cba2. Find a+b.

NS.67.2.5

Two brothers inherit a herd of cattle which they sell for as many dollars per head as there are head of cattle. They then spend all the proceeds to buy sheep at $10 each and one lamb for less than $10. They divide the flock of sheep and one lamb so that each man has the same number of animals. How much did the lamb cost ?

NS.67.3.4

Two externally tangent circles have diameters of 12 units and 8 units respectively. Give the length of their common external tangent.

NS.67.4.4

Twin primes are defined as being a pair of primes such that both p and p+2 are prime numbers. How many such pairs are there between 1 and 100 ?

NS.67.5.6

Factor over the integral domain: $2x^4 - 3x^3 - 5x^2 - 3x + 2$

1967 - 1968

NS.68.1.8

Find the sum of the series: $(1)(2) + (2)(3) + (3)(4) + (4)(5) + \ldots$ to n terms.

Hint: $1^2+2^2+3^2+\ldots+n^2 = (n(n+1)(2n+1))/6$.

NS.68.2.5

Given: Parallelogram ABCD, E is the midpoint of BC. F is a point within BE. Lines AE and DF intersect in G. BF = n and FE = m. The area of triangle EFG = k. Find the area of triangle ADG in terms of m, n, and k.

NS.68.3.6

Solve for x and y:

$$x + y + \sqrt{x+y} = 12$$

$$x^2 + y^2 = 45$$

NS.68.4.6

Given circle O with radius R. Two perpendicular chords AB and CD meet in E. If AE = a, EB = b, CE = c and DE = d, find R^2 in terms of a, b, c, and d.

NS.68.5.5/2

Solve for x:

$$(ax)^2 + abx + acx + bc = 0.$$

1968 - 1969

NS.69.1.7

A rectangular parallelepiped has edges with (integral) lengths x, y, and z. The sum of the lengths of all 12 edges is 76 inches, the sum of the areas of all 6 faces is 192 square inches, and the volume of the solid is 144 cubic inches. Find the length of a diagonal of the parallelepiped.

NS.69.2.4

How many whole numbers between 100 and 15,000 can be expressed in the form $11n^2$ where n represents an integer?

NS.69.3.7

The sum of n terms of the series 3 + 33 + 333 + ... is given by the expression $a(10^n - 1) + bn$. Find the value of a+b.

NS.69.4.4

Write the expression

$x^4 + 6x^3 + 9x^2 - 16$ as the product of a quadratic and two linear factors over the integral domain.

NS.69.5.7

On the equal sides of an isosceles triangle whose sides are 25 inches, 25 inches, and 48 inches, squares are constructed. Find the distance between the centers of the squares.

Additional Answers - Nassau - Suffolk Meet Problems

(Answers begin on next page.)

1966 - 1967 Nassau - Suffolk

67.1 8
67.2 6
67.3 $4\sqrt{6}$
67.4 8
67.5 $(2x^2-3x+2)(x^2-3x+1)$

1967 - 1968 Nassau - Suffolk

68.1 $(n(n+1)(n+2))/3$
68.2 $k(m^2)/(4(m+n)^2)$
68.3 (3,6), (6,3)
68.4 $R^2 = (a^2+b^2+c^2+d^2)/4$
68.5 $x = -c/a, -b/a$

1968 - 1969 Nassau - Suffolk

69.1 13
69.2 33
69.3 1/27
69.4 $(x-1)(x+4)\cdot(x^2+3x+4)$
69.5 31

ANSWERS: PLAY-OFF PROBLEMS and NASSAU-SUFFOLK MEET PROBLEMS

1959 - 1960 Play-Off

NONE. They're all yours. I've run out of time. So you'll just have to ask your Mathlete coach for the right answers. Now isn't that a pernicious suggestion ?!?! And a nasty one too ?

1960 - 1961 Play-Off

61.1 20/3 61.2 15 61.3 12 61.4 $3\sqrt{5}/2$
61.5 160 61.6 1000, 1/100 61.7 4/5
61.8 5, 10/3, 20/9

1961 - 1962 Play-Off

62.1 10 62.2 24 62.3 $1 + \sqrt{2} + \sqrt{3}$
62.4 8 62.5 765

1962 - 1963 Play-Off

63.1 1154 63.2 130 + (10/11) 63.3 60
63.4 18 63.5 0

1964 - 1965 Play-Off

65.1 32 65.2 169 65.3 -1/30 65.4 75
65.5 331 65.6 none (points colinear) 65.7 19/95; 26/65; 49/98

1966 - 1967 Play-Off

67.1 280 67.2 $2 + \sqrt{5}$ 67.3 189 67.4 79
67.5 16

1968 - 1969 Play-Off

69.1 147/30 69.2 $\sqrt{65}$ 69.3 $(3r)/2^{n+1}$ 69.4 8-15-17;
9-40-41; 5-12-13; 20-21-29 69.5 $45\sqrt{3}/8$ 69.SD1 43
69.SD2 194

1965 - 1966 Nassau-Suffolk

66.1 8 66.2 $b\sqrt{6}/3$ 66.3 $(7/9)(10^{2n} - 1)$
66.4 $x^2 + y^2 = 144/25$ 66.5 gain; $200

The rest of the Nassau - Suffolk Meet Answers will be found on the preceding page, namely page 83.

NCIML PROBLEMS 1969 - 1970

70.1.6.12

Express as a fraction a/b, where a and b are relatively prime positive integers, the product

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})(1-\frac{1}{4^2})\cdots(1-\frac{1}{21^2})$$

70.2.7.09

In parallelogram ABCD, AB, BD and DA are 14, 15, and 13 inches long respectively. The length of AC is $\sqrt{k}$ inches. Find k.

70.3.7.02

Let p represent a prime number. Find all integral values of x and y that satisfy $1/(p+x) + 1/(p+y) = 1/p$. Give answers in terms of p and in the form (x,y).

70.4.7.15

In isosceles trapezoid ABCD, AB is a base, the measure of angle B is 45°, and the area of the trapezoidal region, in square inches, is $72(\sqrt{2}+1)$; and BD bisects angle ABC. Find the inch-length of BC.

70.5.5.50

For real numbers that satisfy $x(x-1) \neq 0$, functions f, g, and h have rules $f(x) = 1-x$, $g(x) = 1 - 1/x$, and $h(x) = x/(x-1)$. Express as a polynomial in x, whose leading coefficient (the coefficient of the highest power of x) is 1, the rule of $h(g(f(x)))$.

70.6.6.25

For what positive integral values of p and q are all of the following true? Give answers in the form (p,q).

a) $101 \leq p \leq 104$ b) $205 \leq q \leq 208$
c) $7^p - 3^q$ is a multiple of 10.

70.7.6.06

The vertices of a parallelogram are A,B,C, D: the midpoints of its sides are E, F, G, H; and its diagonals intersect in J. How many triangles are there whose vertices are in the set of the nine named points ?

70.8.7.04

Find all positive integral values of x and y that satisfy $5x^2 - 10xy + 7y^2 = 55$. Give answers in the form (x,y).

70.9.7.01

In square ABCD, M is the midpoint of BC. The perpendicular from B to AM intersects DC in E. The perpendicular from D to AM intersects AM in F. The inch-length of AB is 4. The inch-length of EF is a/b where a and b are relatively prime positive integers. Find a and b and give the answer in the form (a,b).

70.10.7.50

Solve for the real values of x: $|2x - 1| + |x-2| = 4$.

70.11.6.46

The mean (average) of the numbers in a set of 3n numbers is 50. The set is subdivided into two disjoint sets, one containing 2n of the numbers, the other containing the remaining n numbers. The mean of the numbers in the larger subset is twice the mean of the numbers in the smaller. Find the mean of the numbers in the smaller subset.

70.12.7.13

In how many different ways can a convex hexagonal region be dissected into four triangular regions by three diagonals ?

is 230,000 miles, and the earth's mass is 81 times that of the moon, and that both earth and moon are stationary.)

70.24.6.00

The unit length of a plane rectangular coordinate system is 1 cm. Find the area, in square cm., of the region of the square that circumscribes the ellipse whose equation is $x^2/25 + y^2/9 = 1$. (Assume the coordinate axes contain the diagonals of the square.)

70.25.7.05

What is the least value $(x+\sqrt{2}+\sqrt{3})(x+\sqrt{2}-\sqrt{3})(x-\sqrt{2}+\sqrt{3})(x-\sqrt{2}-\sqrt{3})$ can have if x represents a real number ?

70.26.7.56

Consider the set of all 5 digit numerals, each one using all the digits 1, 2, 3, 4, 5 in some order. Assume that they are arranged in order of numerical value starting with least value. What is the 55th numeral in this sequence ?

70.27.[illegible].00

Rectangle PQRS is inscribed in rectangle ABCD, with P in DA, Q in AB, R in BC, and S in CD. The inch-lengths of AB and BC are 6 and 10 respectively. If P is distinct from A and AP < 5, what are all the possible values of AP, in inches ?

70.28.6.55

For what real number values of x will the relationship $0 < x^2 - 4x < 5$ be true?

70.29.6.15

Angle C of triangle ABC is an acute angle. Squares BCDE and ACFG are drawn in the exterior of the triangle. The distance between the centers of the squares is 8 inches. Find FB to the nearest tenth of an inch.

70.30.6.72

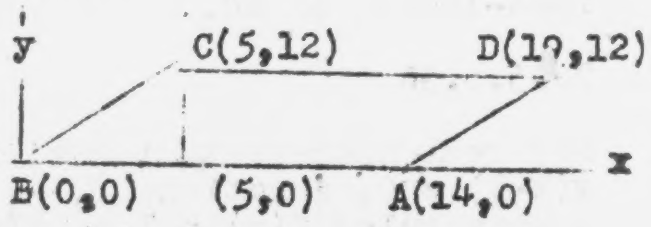
Find a and b in $y = a|x-1| + b|x+1|$, if the equation is satisfied by $(x,y) = (2,2)$ and also by $(x,y) = (-2,-2)$. Give the answer in the form (a,b).

SAMPLE SOLUTIONS 1969 - 1970

70.1.6.12

Note that each factor is itself the difference of two perfect squares: therefore, (1 - 1/4) factors into (1- 1/2)(1+ 1/2) or (1/2)(3/2); (1- 1/9) factors into (1- 1/3)(1+ 1/3) or (2/3)·(4/3). Note that the second factor of the first expression is the multiplicative inverse of the first factor of the second expression. Since this will be the case all the way down the line, the product simplifies to (1/2)(22/21).

70.2.7.09



C(5,12) is known immediately because of the nature of the 13-14-15 triangle. D(19,12) immediately follows. Use the distance formula from there.

70.3.7.02

The simplification of the given relationship should lead you to the relationship $xy = p^2$. Therefore, since p is prime, the question becomes, "How many factorizations of p^2 are there ?" Of course you had best avoid the (-p,-p) solution because of the given form of the problem.

70.13.6.07

Find positive integral values for x, y, and z such that $x < y$, $(3^2 + 4^2 + x^2)$ is the square of an integer, and $3^2+4^2+x^2+y^2=z^2$. Give answers in the form (x,y,z).

70.14.6.30

ABCD is a square. Semi-circles are drawn in the interior of the square, one on each side of the square as diameter. This forms a four-leaf clover whose area is $72(\pi - 2)$ square inches. Find the inch-length of AB.

70.15.7.39

Given: $n = ((p+1)/p)^p$,
$m = ((p+1)/p)^{p+1}$, and
$p = 10$. Find the value of $m^n - n^m$. (That is, find the integral value of the given expression explicitly.)

70.16.7.08

How many positive integers equal to or less than 3,300 are divisible by neither 3 nor 5 nor 11 ?

70.17.7.51

Each edge of a cube is 5 inches long. Three faces of the cube having a common vertex are each subsivided into 25 congruent squares (each one inch on a side). In each of these faces the second and fourth squares along each diagonal are chosen. Starting at the chosen squares, square tunnels are excavated through the cube, perpendicular to its face. What is the volume (in cubic inches) of the solid that is left ?

70.18.7.22

Find the (integral) value of

$1^2 - 2^2 + 3^2 - 4^2 + \ldots + 199^2$, where the dots represent the missing squares that are added and subtracted alternately.

70.19.6.16

The unit length of a plane rectangular coordinate system is 1 inch. The area of the intersection of the regions determined by $x^2 + y^2 \leq 36$ and $y \geq 3$ is $a\pi - b\sqrt{3}$. Find a and b and give the answer in the form (a,b).

70.20.6.66

In a collection of rectangular tiles some have a width of 3 inches and a length of 5 inches. All the others have a width of 7 inches and a length of 11 inches. The number of smaller tiles is a multiple of 5. The number of inches in the sum of the widths of all the tiles is 114. What is the number of inches in the sum of their lengths?

70.21.5.05

What is the smallest positive integral value of x at which $x^2 + x + 41$ is not a prime ?

70.22.7.09

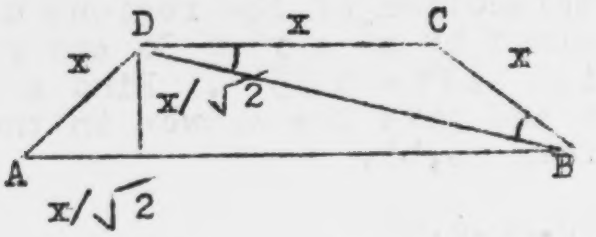
The base of an isosceles triangle is 60 feet long and each leg is 50 feet long. Circles are drawn, with the vertices of the triangle as centers, that are externally tangent to each other in pairs. Find, in feet, the radius of a fourth circle that is externally tangent to each of the other circles.

70.23.6.21

The gravitational attraction between two objects, in appropriate units, is given by the expression kmM/d^2 where m amd M are their respective masses, d is the distance in miles between them and k is a constant. A rocket is shot from the earth to the moon. At how many miles from the moon do the earth and the moon exert equal attractions on the object? (Assume the earth-moon distance

70.4.7.15

Would you believe that triangle DCB is isosceles ???? Just try out the angle measures and see!! Oh boy what a simple problem!!!



$$72(\sqrt{2}+1) = (1/2)(x+(x+2(x/\sqrt{2})))\cdot(x/\sqrt{2})$$

or

$$(1/2)x^2(\sqrt{2}+1) = 72(\sqrt{2}+1) \text{ .}$$

70.5.5.50

Composition of functions:

$g(f(x)) = 1 - 1/(1-x) = x/(x-1)$

And

$h(g(f(x))) = (x/(x-1))/((x/(x-1))-1)$

Simplify.

70.6.6.25

$7^{101} = 7(7^2)^{50} \equiv 7(-1)^{50}$, mod 10

Therefore, $7^{101} \equiv 7$, mod 10.

Also, $7^{102} \equiv 9$, $7^{103} \equiv 3$, $7^{104} \equiv 1$, all modulo 10.

Since $3^{205} = 3(3^2)^{102}$, $3^{205} \equiv 3$, $3^{206} \equiv 9$, $3^{207} \equiv 7$ and $3^{208} \equiv 1$, all modulo 10. Since $9 - 9 = 0$, $7^{102} - 3^{206} \equiv 0$, modulo 10. Go.

70.7.6.06

Combinations problem: ${}_9C_3 = (9!)/(6!)(3!) = 84$ is the number of ways of selecting nine pointes three at a time. But there are 8 line segments consisting of 3 points. $84 - 8 = 76$.

70.8.7.04

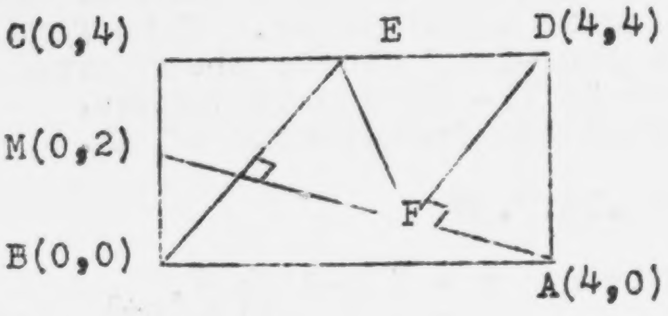
The Clever Way: $5x^2 - 10xy + 5y^2 = 55$; $5(x-y)^2 = 55-2y^2$.

$(x-y)^2 = 11 - (2y^2)/5 > 0$.

Since any integer squared must be greater than zero (with the obvious exception), the latter inequality must be so; and since this is an integer question, $2y^2$ must be divisible by 5. Go.

Fakeroo: Let $y = 1$, try to factor. Let $y = 2$, try to factor. Let $y = 3$, try to factor. ETC.

70.9.7.01



$m_{AM} = -1/2$; $m_{BE} = 2$; L_{BE}: $y = 2x$

$E(x,4)$. Therefore, $E(2,4)$.

L_{AM}: $x+2y = 4$; L_{DF}: $2x-y = 4$.

Therefore, $F(12/5, 4/5)$.

$EF = \sqrt{4/25 + 256/25}$

70.10.7.50

Four possible cases: (+,+), (+,-), (-,+), (-,-). Solve each equation, check each answer, and reject those that don't fit.

70.11.6.46

$S_{3n}/3n = 50$; $S_{3n} = 150\,n$

$S_{2n}/2n = M_2$; $S_{2n} = (M_2)(2n)$

$S_n/n = M_1$; $S_n = n\cdot M_1$; $M_2 = 2\cdot M_1$.

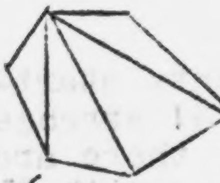
Therefore,

$S_{3n} = 150n = S_{2n} + S_n = 4nM_1 + nM_1$

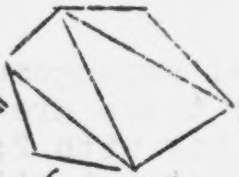
70.12.7.13

Since the hexagonal region is not necessarily regular, each set of triangles obtained by drawing some three diagonals constitutes a different way of dissecting

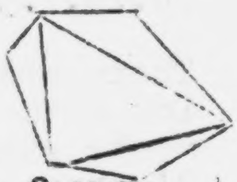
70.12.7.13 the region.



6 ways



6 ways



2 ways .

70.13.6.07

Since $9+16+x^2 * 25+x^2 = n^2$, the x must be 12 because you can only have a 5-12-13 right triangle. And so $13^2+y^2=z^2$ should also lead you immediately to the recognition that the 13-84-85 right triangle is the key. Of course, if you're not familiar with this right triangle, try: $13^2 = z^2-y^2 = (z-y)(z+y) = 169 \cdot 1$. Go.

70.14.6.30

If you draw the figure carefully and give it a careful look, your must see that the clover leaf = 4 semi-circles - square. It's that simple.
Clover-leaf = $4(\pi s^2/8) - s^2 = 72(\pi - 2)$ Go.

70.15.7.39

Right off, $n = (11/10)^{10}$ and $m = (11/10)^{11}$. Then it's strictly a matter of the laws of exponents.

$$m^n = ((11/10)^{11})^{(11/10)^{10}} =$$

$$((11/10)^{11 \cdot 11^{10} \cdot (1/10^{10})} =$$

$$(11/10)^{11^{11} \cdot (1/10^{10})}$$

Likewise, the second term. Note that the problem could be worked out in general without using p = 10.

70.16.7.08

In the given sequence of numbers 2/3 are not divisible by 3; of the remaining set, 4/5 are not divisible by 5; and of the new remaining set, 10/11 are not divisible by 11; so $3300 \cdot 2/3 \cdot 4/5 \cdot 10/11 = 1600$ numbers which are divisible by neither 3, 5, nor 11 .

70.17.7.51

Look at one face: four tunnels, for a volume of 20 in^3. Therefore the three faces give rise to a volume of 60 in^3. But there are 8 intersections which have been counted three times; they should only be counted once. Subtract 16 in^3. Therefore, there is 44 in^3 of hollowness in the cube, and 81 in^3 of solid.

70.18.7.22

Group the series such that

$1^2 + (3^2-2^2) + (5^2-4^2) + \ldots$. Since the second term is the difference of two perfect squares, a s are all the rest, the sequence breaks down into $1 + (3+2) + (5+4) + (7+6) + \ldots$ or $1+5+9+13+17+\ldots$, an arithmetic progression. First plus last times the number of terms divided by two. Go.

70.19.6.16

The central angle of the sector is 120°. Remember the 30-60-90 right triangle; in this case a $3 - 3\sqrt{3} - 6$ right triangle. The area of the sector is $(1/3)(36\pi)$; the area of the triangle is $(3)(3\sqrt{3}) = 9\sqrt{3}$. Go.

70.20.6.66

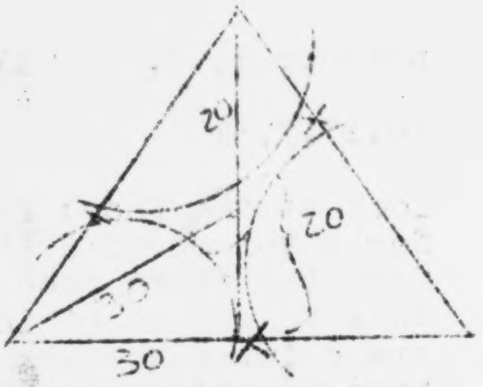
$n \cdot 3 + m \cdot 7 = 114$; $n = 5k$. $15k + 7m = 114$. By fakeroooo, the only possible all positive integer solution is $k = 2$, $m = 12$. Therefore, $5n + 11m = 182$.

70.21.5.05

The expression is a very well traveled generator of (only) prime numbers ... up to a point. Other examples: x^2+x+17 and $x^2-79x+1601$. Be prepared.

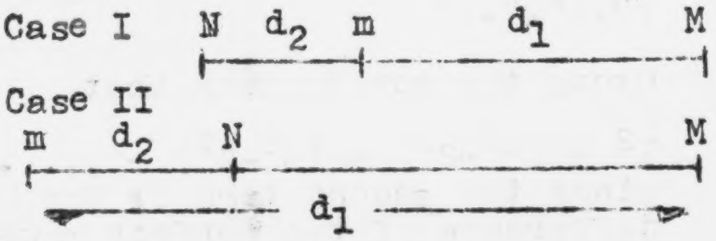
Ans. Seq. Cont: 40; 4; 23,000; 68; -24; 32145; $0 < AP \leq 1$; $-1 < x < 0$ or $4 < x < 5$; 11.3; (-1,1) .

70.22.7.09



x is the radius of the small circle. $(30 + x)^2 = (20 - x)^2 + 30^2$. Go.

70.23.6.21

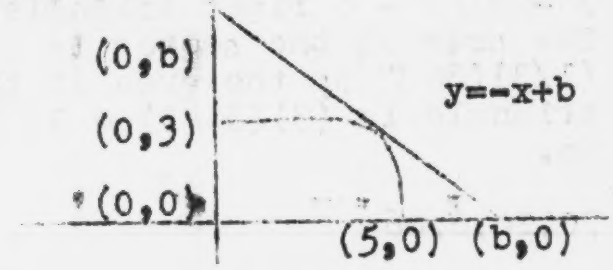


$kmM/d_1^2 = kmN/d_2^2$; m = mass of rocket
M = mass of earth; N = mass of moon. Therefore, M = 81N
$81N/d_1^2 = N/d_2^2$; $9d_2 = d_1$.

$d_1+d_2 = 230{,}000$ or $d_1-d_2 = 230{,}000$

The first relationship yields the right answer; the second, the Charles Harrison answer.

70.24.6.00



$y = -x + b$ is substituted into the given equation yielding the equation $34x^2-50bx+25b^2 -225 = 0$. Since this equation must have equal roots (symmetry), set the discriminant = to 0. $b^2=34$.

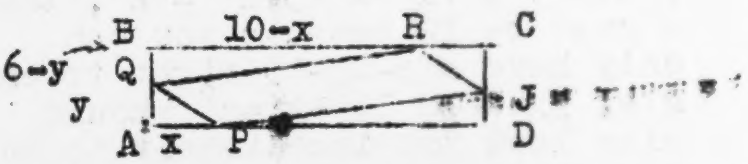
70.[illegible]

Multiply the first and fourth factors together cleverly; i.e., (x+A)(x-A) and ditto the 2nd and 3rd. Result: $x^4 - 10x^2 + 1$, a parabola in x^2. $-b/2a = 5$. Therefore substitute $\sqrt{5}$ to obtain the minimum value of the expression.

70.26.7.56

There are 4! arrangements starting with 1; there are 4! arrangements starting with 2; there are 3! arrangements starting with 31. #55 will be 32145 .

70.27.6.00



Similar triangles: $x/y = (6-y)/(10-x)$. $y^2 -6y + 10x -x^2 = 0$.
The discriminant of this quadratic becomes $9-10x+x^2$, and must be greater than zero. Go.

70.28.6.55

$x^2-4x > 0$ implies $x < 0$ or $x > 4$
$x^2-4x < 5$ implies $-1 < x < 5$.
Merely take the intersection of the three sets.

70.29.6.15

Draw segments CM and CN. Note that triangles CMN and FCB are similar., with a ratio of similitude of $1:\sqrt{2}$. $FB = 8\sqrt{2}$.

70.30.6.72

If the given values for x and y are substituted into the equations, $2 = a+3b$ and $-2 = 3a+b$ result. Solve.

ANSWERS: To add some well deserved confusion to this terribly clear exposition I will merely list the answers as an ORDERED sequence. This way it's tougher to take a SP (Sneak Peak) at an answer before you've given it a good try.

Answers L969 - 1970

11/21; 505; $(p,p),(1,p^2),(p^2,1),(-p^2,-1),(-1,-p^2)$; 12; x; (101,207),(102,206),(103,205),(104,208); 76; (6,5),(4,5); (52,5); -1/3,7/3; 30; 14; (12,84,85); 12; 0; 1600; 81; 19,900; (12,9); 182; (See Prev. Page)

NCIML PROBLEMS 1970 - 1971

NOTATION: 71.1.4.55 MEANS THAT THE YEAR IS 70 - 71, IT IS PROBLEM #1, THE TIME LIMIT WAS 4 MINUTES, AND 55% OF THOSE COMPETING FOUND THE CORRECT SOLUTION.

71.1 4.55

ASSUME THAT FOR THE REAL NUMBERS A, B, C, IN THAT ORDER, SUBTRACTION IS ASSOCIATIVE. WHAT ARE THE POSSIBLE VALUES OF C?

71.2.6.12

EACH EDGE OF A CUBE IS 2 INCHES LONG. THE CENTER OF EACH FACE IS JOINED BY LINE SEGMENTS TO THE CENTERS OF THOSE FACES THAT HAVE AN EDGE IN COMMON WITH IT. THIS DETERMINES 8 TRIANGULAR REGIONS THAT BOUND A SOLID. FIND, IN CUBIC INCHES, THE VOLUME OF THAT SOLID.

71.3.5A (CORRECTED ! ! !) GIVEN:

$(1/x^2) + (1/Y^2) = A\sqrt{2}$, AND $XY = A$ $(A \neq 0)$. $(X + Y)$ EXPRESSED IN TERMS OF A TAKES THE FORM $A^P . \sqrt{2} + AQ$, WHERE P AND Q ARE NUMBERS. FIND P AND Q AND EXPRESS IN THE FORM (P,Q).

71.4.6.14

THE PERIMETER OF AN ISOSCELES TRIANGLE IS THE SQUARE OF THE ALTITUDE TO THE BASE. WHAT IS THE SMALLEST POSSIBLE AREA, OF THIS TRIANGULAR REGION IF BOTH ALTITUDE AND THE AREA ARE POSITIVE INTEGERS?

71.5.4.42

$A = 1+x+x^2$; $B = 1-x+x^2$; $C = 1+x-x^2$; $D = 1-x-x^2$, D CAN BE EXPRESSED AS $(aA+bB+cC)$, WHERE A, B, C ARE NUMBERS. FIND THE VALUES OF A, B, AND C AND EXPRESS YOUR ANSWER IN THE FORM (A,B,C).

71.6.4.32

A MAN TRAVELS M MILES AT THE RATE OF 2 MINUTES PER MILE, AND HE RETURNS TO HIS STARTING POINT, ALONG THE SAME PATH, AT THE RATE OF 2 MILES PER MINUTE. FIND THE AVERAGE RATE IN MILES PER MINUTE FOR THE ENTIRE TRIP. (ANSWER MUST BE EXPRESSED AS A RATIONAL NUMBER IN SIMPLEST FORM.)

71.7.7.02

A STRIP OF UNIFORM WIDTH IS CUT FROM EACH SIDE OF AN EQUILATERAL TRIANGULAR CARDBOARD, LEAVING A SMALLER EQUILATERAL TRIANGULAR CARDBOARD WHOSE AREA IS THREE FOURTHS OF THAT OF THE ORIGINAL. FIND THE RATIO OF THE WIDTH OF THE STRIP TO THE LENGTH OF THE ORIGINAL SIDE, EXPRESSED AS A DECIMAL FRACTION, CORRECT TO THE NEAREST HUNDREDTH.

71.8.5.10

LET F BE THE FUNCTION THAT INCREASES A NUMBER BY B, $B \neq 0$ AND G THE FUNCTION THAT CALLS FOR THE RECIPROCAL OF A NUMBER. FOR EACH OF CERTAIN VALUES OF B THE EQUATION $F(G(X)) = G(F(X))$ HAS A UNIQUE SOLUTION. FIND THESE VALUES OF B.

71.9.7.22

IN A CERTAIN RECTANGULAR COORDINATE SYSTEM WHOSE UNIT DISTANCE IS 1 INCH, THREE LINES HAVE RESPECTIVE EQUATIONS OF $Y = 0$, $Y = 1$, AND $Y = 3$. FIND, IN SQUARE INCHES, THE THE AREA OF THE REGION OF AN ISOSCELES RIGHT TRIANGLE WHOSE RIGHT ANGLE VERTEX IS ON THE LINE $Y = 1$, AND WHOSE OTHER VERTICES ARE ON THE OTHER TWO LINES, THAT IS, ONE ON THE LINE $Y = 0$ AND THE OTHER ON THE LINE $Y = 3$. GIVE YOUR ANSWER IN THE FORM A/B, WHERE A AND B ARE RELATIVELY PRIME INTEGERS.

71.10.4.41

LET $A_0=1$, $A_1=3$, AND $A_{N+2}=2A_{N+1}+3A_N$. EXPRESS THE VALUE OF A_{50} AS A POWER OF 9.

71.11.5.19

THERE ARE SEVEN THREE-DIGIT NUMERALS THAT REPRESENT SQUARES OF INTEGERS AND ARE SUCH THAT WHEN EACH IS DIVIDED BY THE SUM OF ITS DIGITS, THE QUOTIENT IS ALSO THE SQUARE OF AN INTEGER. FOUR OF THESE NUMERALS ARE 100, 144, 400, 900. WHAT ARE THE OTHER THREE???

71.12.7.04

TWO CIRCLES MEET AT A AND D. A LINE THROUGH D CUTS THE CIRCLES IN POINTS B AND C SUCH THAT D BISECTS BC. FIND THE INCH MEASURE OF THE RADIUS OF THE SMALLER CIRCLE IF THE INCH MEASURES OF AB, AD, AND AC ARE $\sqrt{7}$, 3, $\sqrt{19}$, RESPECTIVELY. EXPRESS THE ANSWER IN THE FORM $\sqrt{A/B}$ WHERE A AND B ARE RELATIVELY PRIME INTEGERS.

71.13.5.36

THE SUM OF N NUMBERS IS S. EACH MEMBER OF THE SET IS DECREASED BY 4, EACH RESULTING DIFFERENCE IS THEN MULTIPLIED BY 4, AND FINALLY EACH PRODUCT IS INCREASED BY 4. THE SUM OF THE FINAL RESULTS IS (AS + BN). FIND A AND B AND EXPRESS YOUR ANSWER IN THE FORM (A,B).

71.14.6.01

IN ISOSCELES TRAPEZOID ABCD, AB IS THE LONGER BASE. THE INCH LENGTHS OF AC AND AB ARE $\sqrt{468}$ AND 23 RESPECTIVELY, AND THE LENGTH OF AD, DC, AND CB ARE THE SAME. FIND, IN SQUARE INCHES, THE AREA OF THE TRAPEZOIDAL REGION.

71.15.6.09

FIND ALL POSITIVE INTEGERS X ANY Y, IF ANY, SUCH THAT $XY-X^2+6X-Y=11$. GIVE YOUR ANSWER(S) IN THE FORM (X,Y).

71.16.6.14

THE INFINITE DECIMAL $X-.X_0X_1X_2X_3\ldots$ IS FORMED AS FOLLOWS: TAKE X_0-3, THEN FOR $N \geq 1$ TAKE THE LEAST POSITIVE REMAINDERS UPON DIVIDING $X_0+X_1+X_2+\ldots+X_N-1$ BY 9. FIND THE VALUE OF X, EXPRESSED AS A FRACTION WHOSE TERMS ARE RELATIVELY PRIME POSITIVE INTEGERS.

71.17.6.02

IN RIGHT TRIANGLES ABC AND DEF, FIVE PARTS OF TRIANGLE ABC ARE CONGRUENT TO FIVE PARTS OF TRIANGLE DEF, BUT THE TRIANGLES ARE NOT CONGRUENT. FIND, TO THE NEAREST TENTH, THE RATIO OF THE LENGTH OF THE HYPOTENUSE OF TRIANGLE ABC TO THE LENGTH OF ITS SHORTER LEG.

71.18.6.79

ONE CANDLE IS TWICE AS LONG AS ANOTHER. THE LONGER CANDLE BURNS DOWN IN 4 HOURS, BUT THE SHORTER BURNS DOWN IN 6 HOURS. THEY ARE LIT AT THE SAME TIME. IN HOW MANY HOURS WILL THEY (FIRST) HAVE THE SAME LENGTH?

71.19.6.03

IN TRIANGLE ABC THE BISECTOR OF ANGLE C INTERSECTS AB IN D. IF THE MEASURE OF ANGLE C IS 120° AND THE INCH-MEASURES OF AC AND BC ARE 6 AND 8, FIND THE INCH-MEASURE OF CD, EXPRESSEDIN THE FORM A/B, WHERE A AND B ARE RELATIVELY PRIME INTEGERS.

71.20.7.02

SOLVE FOR X AND Y.

$$((1+\sqrt{5})/2)^2X+((1-\sqrt{5})/2)^2Y=2$$

$$((1+\sqrt{5})/2)X+((1-\sqrt{5})/2)Y=1$$

EXPRESS THE SOLUTION IN THE FORM (X,Y) WHERE X AND Y HAVE THE FORM $(A+B\sqrt{5})/C$, WITH A, B, AND C AS INTEGERS.

71.21.7.11

HOW MANY INTEGERS ARE THERE BETWEEN 40,001 AND 50,000 THAT ARE SQUARES OR CUBES, BUT NOT BOTH?

71.22.6.15

EACH SIDE OF AN EQUILATERAL TRIANGLE IS TRISECTED, AND ON THE MIDDLE SECTION OF EACH SIDE, AN EQUILATERAL TRIANGLE IS DRAWN IN THE EXTERIOR OF THE TRIANGLE. THE MIDDLE SECTIONS ARE THEN ERASED LEAVING A SIXPOINTED STAR. EACH SIDE OF THE STAR IS TRISECTED AND EQUILATERAL TRIANGLES ARE DRAWN IN THE EXTERIOR OF THE FIGURE ON EACH MIDDLE SECTION. THE MIDDLE SECTIONS ARE THEN ERASED. THIS OPERATION IS PERFORMED TWO MORE TIMES. FIND, IN INCHES, THE PERIMETER OF THE LAST FIGURE IS EACH SIDE OF THE ORIGINAL TRIANGLE MEASURES 3 INCHES. EXPRESS ANSWER IN THE FORM A/B, WHERE A AND B ARE RELATIVELY PRIME POSITIVE INTEGERS.

71.23.6.20

IN THE NON-TERMINATING DECIMAL. .123450123450012345000... THE NUMBER OF ZEROS IN EACH REPETITION INCREASES BY ONE. WHAT IS THE DIGIT IN THE 1700TH PLACE?

71.24.7.07

IN ACUTE TRIANGLE ABC, ANGLE A EQUALS 45 DEGREES, ALTITUDE AD MEASURES 10 INCHES, AND BE AND CF ARE ALTITUDES. FIND THE PERIMETER (IN INCHES) OF TRIANGLE DEF.

71.25.7.40

FIND THE VALUE OF $(x-3y)$ IF X AND Y ARE REAL NUMBERS AND

$$x^2 + y^2 = 4x - 6y - 13.$$

71.26.8.24

IN THE EQUATIONS

$$(ST)^P = S\ L\ O\ E\ S$$

$$(TS)^P = E\ E\ L\ P\ I\ T$$

ALL LETTERS REPRESENT DIGITS IN DECIMAL NUMERALS. DIGITS ARE DIFFERENT IF AND ONLY IF THEIR LETTERS ARE DIFFERENT. FIND THE NUMBER REPRESENTED BY ST.

71.27.6.29

THE AREA OF A CIRCULAR REGION IS DOUBLED WHEN ITS RADIUS R IS INCREASED BY N. EXPRESS TO THE NEAREST TENTH, THE RATIO OF R TO N.

71.28.5.25

ASSUMING THAT IS EXITS AND IS POSITIVE, EXPRESS

$$\sqrt[3]{5\sqrt[3]{5\sqrt[3]{5\ldots}}}$$

(WHERE ... MEANS: "AND SO ON ENDLESSLY") AS A DECIMAL NUMERAL, TO THE NEAREST TENTH.

71.29.6.22

A ONE-INCH LINE SEGMENT IS MUTILATED IN STAGES AS FOLLOWS: IN THE FIRST STAGE THE MIDDLE THIRD OF THE SEGMENT IS DISCARDED. IN THE SECOND STAGE THE MIDDLE THIRD OF [illegible] THE TWO [illegible] IS DISCARDED. IN EACH SUCCEEDING STAGE ~~THE~~ MIDDLE THIRD OF EACH REMAINING SEGMENT IS DISCARDED. AFTER THE (N-1)TH STAGE THE SUM OF THE LENGTHS OF THE REMAINING SEGMENTS IS GREATER THAN .01 INCH, BUT AFTER THE NTH STAGE THE SUM OF THE REMAINING SEGMENTS IS LESS THAN .01 INCH. FIND N. (APPROXIMATIONS ARE AS FOLLOWS: LOG 2 = .301, LOG 3 = .477).

71.30.6.05

HOW MANY TIMES DOES 2 APPEAR AS A FACTOR IN THE SMALLEST INTEGER THAN IS GREATER THAN $(4 + 2\sqrt{3})^6$???

SAMPLE SOLUTIONS 1970-- 1971

71.1.4.55

$A-(B-C) = (A-B)-C$ THEREFORE,
$A-B+C = A-B-C$. $2C = 0$.

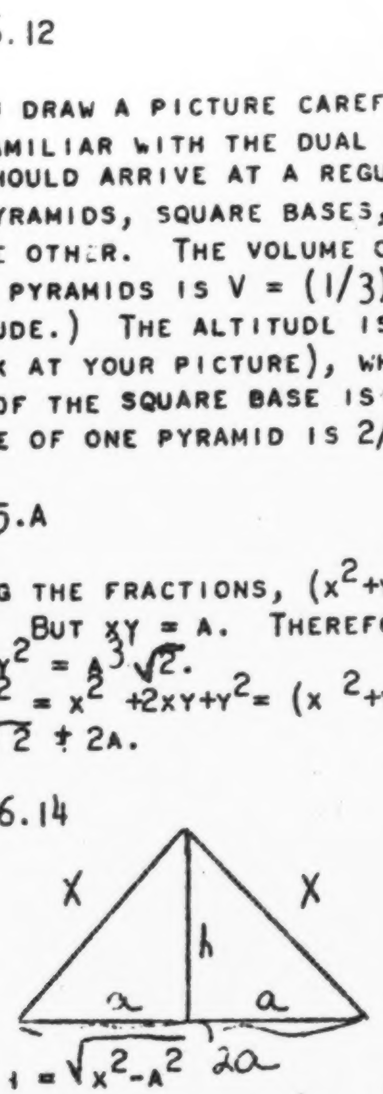
71.2.6.12

IF YOU DRAW A PICTURE CAREFULLY (OR ARE FAMILIAR WITH THE DUAL OF THE CUBE), YOU SHOULD ARRIVE AT A REGULAR OCTAHEDRON. TWO PYRAMIDS, SQUARE BASES, ONE ON TOP OF THE OTHER. THE VOLUME OF ONE OF THESE PYRAMIDS IS $V = (1/3)$(BASE x ALTITUDE.) THE ALTITUDE IS OBVIOUSLY 1 (LOOK AT YOUR PICTURE), WHEREAS THE SIDE OF THE SQUARE BASE IS $\sqrt{1^2 + 1^2} = \sqrt{2}$. VOLUME OF ONE PYRAMID IS $2/3$.

71.3.5.A

ADDING THE FRACTIONS, $(x^2+y^2)/(xy)^2 = A\sqrt{2}$. BUT $xy = A$. THEREFORE,
$x^2 + y^2 = A^3\sqrt{2}$.
$(x+y)^2 = x^2 + 2xy + y^2 = (x^2 + y^2) + 2xy$
$= A^3\sqrt{2} + 2A$.

71.4.6.14



1). $H = \sqrt{x^2-A^2}$
2). $P = 2x + 2A$ 3). $H^2 = x^2-A^2$

THEREFORE, $2(x+A) = x^2 - A^2$
$2 = x-A$; $x = A+2$.

$$\text{AREA} = (1/2)(2A)\sqrt{x^2-A^2}$$
$$= A\sqrt{(A+2)^2 - A^2} = 2A\sqrt{A+1}$$

SINCE $A \neq 0$, $A = 3$. GO.

71.5.6.42

$1-x-x^2 = A+Ax+Ax^2+B-Bx+Bx^2+C+Cx-Cx^2$
$= (A+B+C) + (A-B+C)x) + (A+B-C)x^2$

EQUATING LIKE COEFFIENENTS:

1). $A+B+C = 1$ 1) AND 2) TOGETHER IMPLIES
2). $A-B+C = -1$
3). $A+B-C = -1$ $A+C = 0$

2). AND 3) TOGETHER IMPLY $2A = -2$.

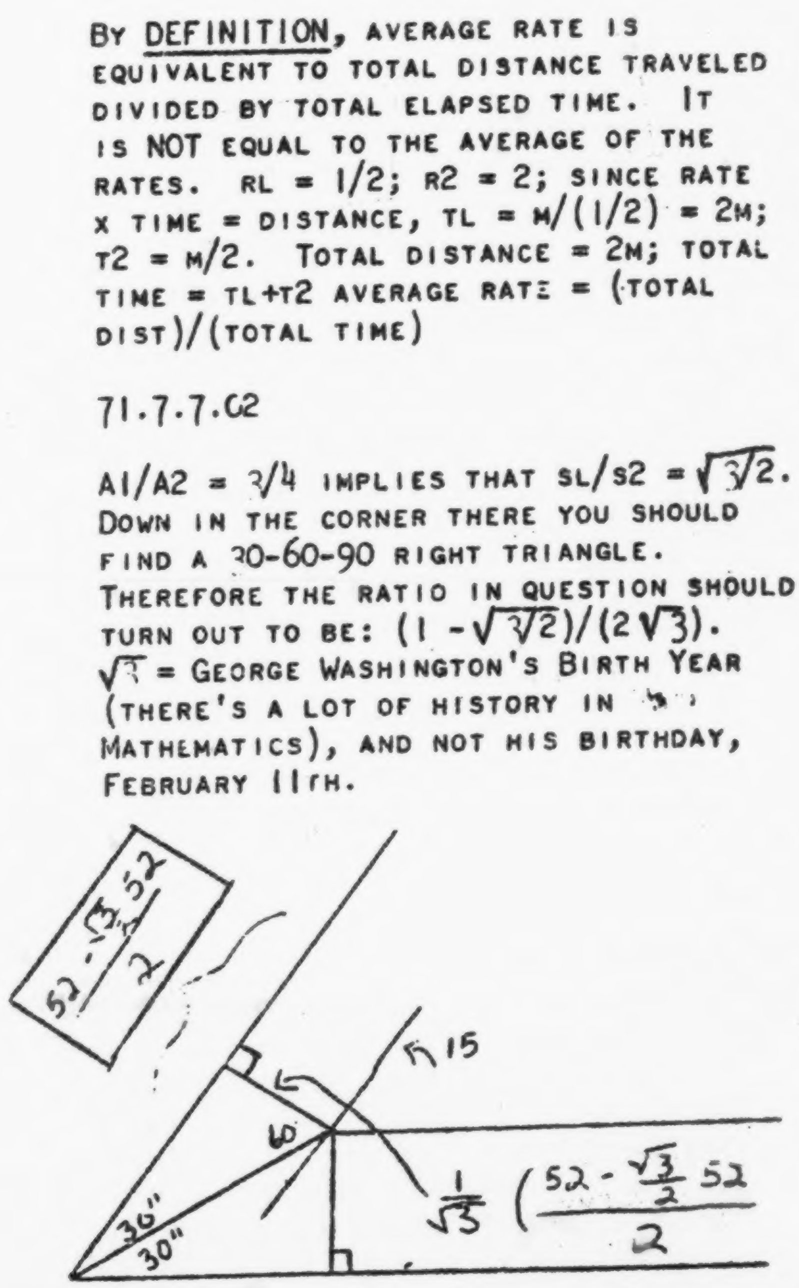
THEREFORE, $C = 1$ AND FROM EITHER 1) OR 2) OR 3), $B = 1$.

71.6.4.22

BY DEFINITION, AVERAGE RATE IS EQUIVALENT TO TOTAL DISTANCE TRAVELED DIVIDED BY TOTAL ELAPSED TIME. IT IS NOT EQUAL TO THE AVERAGE OF THE RATES. $RL = 1/2$; $R2 = 2$; SINCE RATE x TIME = DISTANCE, $TL = M/(1/2) = 2M$; $T2 = M/2$. TOTAL DISTANCE = 2M; TOTAL TIME = TL+T2 AVERAGE RATE = (TOTAL DIST)/(TOTAL TIME)

71.7.7.02

$A1/A2 = 3/4$ IMPLIES THAT $SL/S2 = \sqrt{3/2}$. DOWN IN THE CORNER THERE YOU SHOULD FIND A 30-60-90 RIGHT TRIANGLE. THEREFORE THE RATIO IN QUESTION SHOULD TURN OUT TO BE: $(1 - \sqrt{3/2})/(2\sqrt{3})$. $\sqrt{3}$ = GEORGE WASHINGTON'S BIRTH YEAR (THERE'S A LOT OF HISTORY IN MATHEMATICS), AND NOT HIS BIRTHDAY, FEBRUARY 11TH.

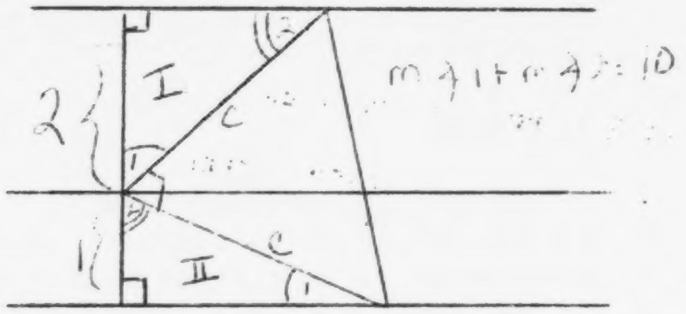

$\frac{52 - \frac{\sqrt{3}}{2}52}{2}$
15
60
30"
30"
$\frac{1}{\sqrt{3}}\left(\frac{52 - \frac{\sqrt{3}}{2}52}{2}\right.$

71.8.5.10

THE IMPORTANT WORD HERE IS UNIQUE. $F(x) = x+B$; $G(x) = 1/x$. $F(G(x)) = (1/x)+B$; $G(F(x)) = 1/(x+B)$. EQUATING THE TWO YIELDS THE RELATIONSHIP $x^2 + Bx + B = 0$. OR $x = (-B + \sqrt{B^2-4})/2$. NOW X WILL ONLY BE UNIQUE WHEN THE DISCRIMINANT IS ZERO (0). THEREFORE, VOILA.

71.9.7.22

HERE I DIDN'T LOOK BEFORE I LEPT, MADE A GOOD ANALYSIS WITH A BAD ALGEBRA MISTAKE, AND GOT NOWHERE. HOWEVER: MERELY NOTE THAT (I.E., OBSERVE) TRIANGLES I AND II ARE CONGRUENT. OY VEY. $c = \sqrt{5}$ AND QED.



71.10.4.41

HARRY'S EASY PROBLEM: ASSUMING YOU'VE SEEN RECURSIVELY DEFINED SEQUENCES BEFORE! TO GET A2 YOU LET N = 0; THEREFORE, $A_2 = 2A_1 + 3A_0 = 6 + 3 = 9$. AND $A_3 = 2A_2 + A_1 = 18 + 9 = 27$. THE SEQUENCE LOOKS LIKE: $(1,3,9,27,81,243,\ldots,3^N,\ldots)$ GO.

71.11.5.19

NOW I DON'T WANT TO BRAG, BUT I HAPPEN TO KNOW THE PERFECT SQUARES FROM 1 TO 3.. AND THEN SOME. I WROTE EM DOWN AND WENT TO WORK. NO SWEAT. THEORY ?? WELL THE DIVISOR WAS 9 IN EACH CASE, BUT WHO KNOWS WHY ????????

71.12.7.04

WAY OUT, WAY OUT. I REALLY DOUBT THAT ANY OF THE ANSWERS OBTAINED WERE ANYTHING BUT GUESSES. MR. JOE HOLBROOK OF FREEPORT SEZ I CAN USE JEB STEWART'S THEOREM (CHARGE!) AND THEN FRANK PTOLEMY, RALPH CEVA AND MAXINE MENELAUS DOES THE JOB. GOOD GRIEF. AL STONE AND I BOTH GO FOR THE LAW OF COSINES IN TRIANGLE ACDB; WHICH IS TO SAY, IN TRIANGLES CDA AND BDA. THE TWO ANGLES AT D ARE SUPPLEMENTARY; AND THE COSINE OF X AND 180-X ARE MERELY ADDITIVE INVERSES OF ONE ANOTHER. USE THE FORM $\cos x = (A^2 + B^2 - C^2)/(2AB)$ TWICE AND YOU'LL HAVE THE LENGTH OF BD AND A SPECIAL CASE OF STEWART'S THEOREM. BD EQUALS CD EQUALS 2. NOW, IF YOU ARE READING THIS, AND STILL HAVEN'T DRAWN AND LABELED A DIAGRAM, STOP BEING SO LAZY (LIKE ME) AND OBSTINAEE, AND GET BUSY. I MADE UP A NICE PICTURE. TAKE THE VALUE OF DB AND PLUG BACK INTO THE COSINE FORMULA: YOU SHOULD GET 1/2. AL SONE DID. AH SO! ANGLE ADB MEASURES 60 DEGREES. ARC AB MEASURES 120 DEGREES, AS DOES ITS CORRESPONDING CENTRAL ANGLE. DRAW APPROPRIATE RADII, SEE THE 30-60-90, AND QED.

71.13.5.6

ONE TERM: $4(A_i - 4) + 4 = 4A_i - 12$. THERE ARE N-12'S, AND THE 4 FACTORS OUT OF EACH OF THE OTHER TERMS. GO.

71.14.6.01

YOU (YOU) DRAW THE PICTURE. CALL ALTITUDE FROM C TO BASE AB D CE; CALL ITS MEASURE H. TWO PYTHAGOREAN RELATIONSHIPS, TRIANGLES ACE AND BCE. $AE = 23-x$, WHERE $EB = x$. $DC = BC = 23 - 2x$. GO. VERY CAREFULLY.

OR. FAKE IT. BREAK THE 23 UP INTO TWO PARTS. HARRY DIDN'T GIVE ANY RADICAL FORM, SO YOU'RE GOING TO BE DEALING WITH INTEGERS. TRY 20 AND 3. 20 SQUARED FROM 468 YIELDS 68, A NO-PERFECT SQUARE. TRY ANOTHER. HOW ABOUT 18 AND 5 ??? 18 SQUARED FROM 46 YIELDS 144. HOLY COW.

71.15.6.09

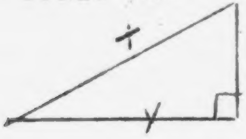
JOE HOLBROOK TELLS ME THAT THIS WAS HARRY'S GIFT PROBLEM. HE SEZ JUST SOLVE FOR Y: $Y = (11 - 6x + x^2)/(x-1)$. LET X TAKE ON VALUES, THE POSITIVE INTEGER JAZZ TAKES CARE OF THE REST. GO.

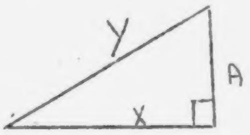
71.16.6.14

$X_0 = 3$. THE RULE SEZ THAT FOR $X = 1$, SUM UP THE SERIES THROUGH $X_{1-1} = X_0$; THEN DIVIDE $X_0 = 3$ BY 9 AND TAKE THE REMAINDER AS YOU VALUE FOR X_1. THEREFORE, $X_1 = 3$. X_2 IS THE REMAINDER OBTAINED WHEN 9 IS DIVIDED INTO THE SUM $X_0 + X_1$, OR 6. THEREFORE, $X_2 = 6$. X_3 IS OBTAINED BY TAKING THE REMAINDER WHEN 9 IS DIVIDED INTO THE SUM $3 + 3 + 6$; THEREFORE $X_3 = 3$. AND GO. $X = .3363636363636363636$ GO.

71.17.6.02

SINCE SSS YIELDS CONGRUENCE, ONE PAIR OF SIDES MUST BE UNEQUAL.





$X/A = Y/X = Z/Y$: $X^2 = AY$

OR $A^2 + X^2 = Y^2$

$A + AY = Y^2$

QUADRATIC IN Y: $Y = \frac{(1 + \sqrt{5})}{2}$ A

71.18.6.79

MAA SPECIAL. FAKEROO WILL SEE YOU THROUGH. ALGEBRA ??? LET X = TIME IN QUESTION.

$2L - (x/4)(2L) \div L - (x/6)(L)$

I.E., EQUATE THE LENGTH OF CANDLE LEFT.

71.19.6.03

DRAW THIS ONE YOURSELF, PLEASE. GET INVOLVED, PLEASE. DID YOU KNOW THAT AN ANGLE BISECTOR DIVIDES THE SIDE IT INTERSECTS INTO THE SAME PROPORTION AS THE TWO SIDES OF THE ANGLE IT BISECTS??? (WHAT HE SAY ???) CA/CB = DA/DB

THEREFORE, AD = 3Y AND BD = 4Y.

LAW OF COSINE (TWICE):

$9Y^2 = x^2 + 36 - 12x(1/2)$

$16Y^2 = x^2 + 64 - 16x(1/2)$ GO ! ! !

OR BRUCE SHECTER STYLE

AREA OF TRIANGLE ADC PLUS AREA OF TRIANGLE DCB MUST EQUAL THE AREA OF TRIANGLE ABC. AREA OF A TRIANGLE EQUALS $\frac{1}{2}AB\sin(C)$. THEREFORE,

$$\tfrac{1}{2}\,6x\,\sin(60^\circ) + \tfrac{1}{2}\,8x\,\sin(60^\circ) = \tfrac{1}{2}\cdot 6\cdot 8\cdot \sin(120^\circ)$$

SINCE SIN 60° = SIN 120°, THE EXPRESSION QUICKLY REDUCES TO $6x + 8x = 48$. TOUGH PROBLEM.

71.20.7.02

JUST GO. SUGGESTIONS. USE DETERMINANTS OR INVERSE MATRICES (WHICH IS WHAT I DID AND MADE IT). ALSO, YOU MIGHT HAVE LABELED THE COEFFICIENTS AS A^2, B^2, A AND B, AND SOLVED THE PAIR GENERALLY AND THEN DO ALL THE ARITHMETIC IN ONE SWELL FOOP.

71.21.7.11

$201^2 = 40,200$; $223^2 = 49,729$; $35^3 = 42,875$; $36^3 = 46,656$, BUT $36^3 = 4^3 \cdot 9^3 = 6^3 \cdot 6^3$

THEREFORE, TOSS OUT THE 216. AND THE 36.

71:22.6.15

AFTER INVESTIGATION, EITHER OF TWO FACTS SHOULD APPEAR RELATIVE TO THE FIGURE:

1. EACH PERIMETER IS 4/3 OF THE PRECEDING PERIMETER. THEREFORE, 9 4/3 4/3 4/3 4/3 = 256/9

OR

2. THE NUMBER OF SIDES IS INCREASING BY A FACTOR OF 4; WHILE THE LENGTH OF EACH SIDE IS MULTIPLIED BY A FACTOR OF 1/3. AS ABOVE.

71.23.6.20

USING THE SUM OF AN ARITHMETIC SEQUENCE:

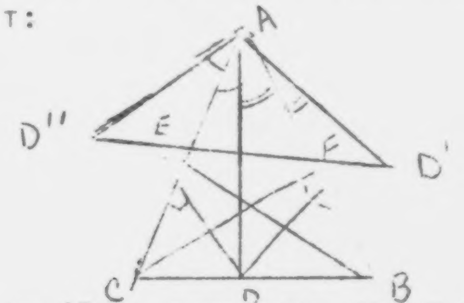
$(5+1) + (5+2)\ (5+3) + \ldots + (5+N) < 1700$

$5N + (N+1)\ (N/2) < 1700$; $N < 54$.

N=53 YIELDS THE VALUE 1696. THEREFORE, THE 97TH TERM IS 1, ETC.

71.24.7 07

TRIANGLE DEF IS CALLED THE PEDAL TRIANGLE OF TRIANGLE ABC. SEAN FAGNANO, AN IRISH MATHEMATICIAN, POSED THE PROBLEM OF FINDING THE TRIANGLE OF MIN MUM PERIMETER THAT CAN BE INSCRIBED IN A GIVEN ACUTE ANGLED TRIANGLE. THE PEDAL TRIANGLE DOES THE TRICK. THE TECHNIQUE OF THE PROOF (COXETER, INTO TO GEOM; WILEY, P.20) YIELDS THE SOLUTION OF THIS PROBLEM. TO WIT:



SEGMENT AD IS REFLECTED ABOUT SIDES AB AND AC OF TRIANGLE ABC YIELDING CONGRUENT SEGMENTS AD" AND AD'. PLEASE NOTE THAT ANGLES CAD AND BAD ARE ALSO REPRODUCED. THEREFORE, ANGLE D"AD' IS A RIGHT ANGLE. DOES SEGMENT D"D' PASS THROUGH POINTS E AND F ?? YOU BET YOUR BIPPY ! (I WONDER WHY? COULD THAT REFLECTION HAVE SOMETHING TO DO WITH IT?) LIKEWISE, D"E' AND D'F ARE CONGRUENT TO SEGMENTS DE AND DF RESPECTIVELY. POTENT TRANSFORMATION, THOSE REFLECTIONS. THERE D"D' IS EQUAL TO THE PERIMETER OF THE PEDAL TRIANGLE DEF.

I UNDERSTAND THAT IF YOU WERE TO STROKE A BILLIARD BALL ALONG SIDE DE OF THE PEDAL TRIANGLE (I.E., IN THE DIRECTION DE), IT WOULD FOLLOW THE PATH OF THE PEDAL TRIANGLE INDEFINITELY. (FRICTION NOT-WITH-STANDING).

SNEAKY PETE WAY: LET THE ANGLES OF ACUTE TRIANGLE ABC MEASURE 45, 45.001 AND 89.999. NOW TRY THE PROBLEM.

71.25.7.40

AFTER BRINGING THE LINEAR TERMS OVER TO THE LHS AND COMPLETING THE SQUARES, YOU SHOULD HAVE: $(x-2)^2 + (y+3)^2 = 0$. THE ONLY WAY THE SUM OF TWO POSITIVE NUMBERS CAN BE ZERO IS NEVER. BY GOLLY, THEY MUST BOTH BE ZERO. x EQUALS 2, AND y EQUALS -3.

71.26.8.24

BY INSPECTION, T MUST BE GREATER THAN S. AND P MUST BE 3 OR 4. IF P EQUALS 3, THEN ST EQUALS 37 OR 28 SINCE T^3 MUST END WITH S. I.E., 7^3 EQUALS 343 AND 8^3 EQUALS 512. P EQUAL 4 YIELDS NO POSSIBILITIES. 37 DOESN'T WORK.

71.27.6.29

$A1 = \pi R^2$. $A2 = \pi (R + N)^2$. $A2 = 2\,A1$

THEREFORE: $2R^2 = (R + N)^2 = R^2 + 2RN + N^2$.

$R^2 - 2RN - N^2 = 0$. $R = \dfrac{2N + \sqrt{4N^2 + 4N^2}}{2}$

$R/N = 1 + \sqrt{2}$.

71.28.5.25

IF (I SAY IF) YOU KNOW WHAT'S HAPPENING HERE, THIS IS THE GIFT OF THE YEAR.

LET x EQUAL THE MONSTOSITY IN QUESTION:

$$x = \sqrt[3]{5 \sqrt[3]{5 \sqrt[3]{5 \cdots}}}$$

NOW. IF YOU WERE TO ERASE (ERASE) THE FIRST CUBE ROOT SIGN AND THE 5 YOU WOULD STILL (STILL!) SEE x. THEREFORE, REPLACE EVERYTHING TO THE RIGHT OF THE FIRST CUBE ROOT SIGN AND THE 5 WITH x. LIKE SO:

$$x = \sqrt[3]{5 \cdot x}\,.$$ THREFORE,

$x^3 = 5x$. POSITIVE ROOT:

71.29.6.22

BY HOOK OR CROOK YOU SHOULD ARRIVE AT THE FOLLOWING SEQUENCE:

$(2/3,\ 4/9,\ 8/27,\ \ldots\ 2^N/3^N)$

WHEN WILL $2^N/3^N < .01$????

$N\ (\log 2 - \log 3) < \log 10^{-2}$?

$N > 2/.176$ FIGURE IT OUT

71.30.6.05

$$(4 + 2\sqrt{3})^6 = 2^6\,(2 + \sqrt{3})^6$$

SO WHO KNOWS WHETHER OR NOT

$(2 + \sqrt{3})^6$ IS BETWEEN AN ODD AND AN EVEN, OR BETWEEN AN EVEN AND AN ODD ??? SO COMPUTE IT, AND FIND OUT ! ! !

OUR AUTHOR, HARRY SITOMER, GIVES THIS PIECE OF INFO:

"THAT INTEGER IS

$$(4+2\sqrt{3})^6 + (4-2\sqrt{3})^6 =$$
$$2^6((2+\sqrt{3})^6 + (2-\sqrt{3})^6) = \ldots "$$

AND THERE YOU HAVE IT: 1970-71

ANSWERS

71.1	0
71.2	4/3
71.3	(3,2)
71.4	12
71.5	(-1,1,1)
71.6	4/5
71.7	.04 OR 4/100
71.8	± 2
71.9	5/2
71.10	9^{25}
71.11	225, 324, 441
71.12	7/3
71.13	(4. 12)
71.14	216
71.15	(2,3), (3,1), (4,1), (7,3)
71.16.	37/100
71.17	1.6 I.E. $(1+\sqrt{5})/2$
71.18	3
71.19	24/7
71.20	$(5+\sqrt{5})/10$, $(5-\sqrt{5})/10$
71.21	23
71.22	256/9
71.23	4
71.24	$10\sqrt{2}$
71.25	11
71.26	28
71.27	2.4
71.28	2.2
71.29	12
71.30	7